A Hotelling-Downs Game for Strategic Candidacy with Binary Issues*

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Abstract

In a pre-election period, candidates may, in the course of the public political campaign, adopt a strategic behavior by modifying their advertised political views, to obtain a better outcome in the election. This situation can be modeled by a type of strategic candidacy game, close to the Hotelling-Downs framework, which has been investigated in previous works via political views that are positions in a common one-dimensional axis. However, the left-right axis cannot always capture the actual political stances of candidates. Therefore, we propose to model the political views of candidates as opinions over binary issues (e.g., for or against higher taxes, abortion, etc.), implying that the space of possible political views can be represented by a hypercube whose dimension is the number of issues. In this *binary strategic candidacy* game, we introduce the notion of local equilibrium, broader than the Nash equilibrium, which is a stable state with respect to candidates that can change their view on at most a given number of issues. We study the existence of local equilibria in our game and identify, in the case of two candidates, natural conditions under which the existence of an equilibrium is guaranteed. To complement our theoretical results, we provide experiments to empirically evaluate the existence of local equilibria and their quality.

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Keywords: Strategic candidacy, Hotelling-Downs model, Local equilibrium

1 Introduction

Strategic voting [2] is a major topic of interest and has been widely studied in Computational Social Choice [3] and Algorithmic Game Theory [4]. While strategic behavior is typically imputed to voters, candidates can also manipulate in real elections. Strategic candidacy [5] occurs when a candidate may strategize by withdrawing from the election in order to obtain a better outcome. Another perspective by which a candidate can be strategic, is to exhibit an insincere political stance [6, 7]. Instead of presenting themselves truthfully, such candidates adopt a dishonest political position whenever it is beneficial.

In order to model the political stance taken by candidates, most papers use a one-dimensional axis to describe the left-right axis of the political spectrum, and study the existence of equilibria in this context (see, e.g., [7]). However, such left-right representation of the political spectrum fails to capture the complexity of current political debates. Benoit and Laver [8] claim that "*this drastically oversimplified notion of a 'left–right dimension' refers to potentially separable issues [...] Indeed, it is very common to need more than one dimension to describe key political differences*" (see also [9]).

A more accurate perspective to describe candidates' positions in the political spectrum can be to consider a list of *issues* on which each candidate is either "in favor" or "against" (e.g., for or against higher taxes, euthanasia, abortion, etc.). Indeed, many complex political opinions can be decomposed into binary opinions on issues, if the decomposition is refined enough. This modeling of the political spectrum can be represented by a hypercube whose dimension is the number of issues: each vertex (or *position*) in the hypercube represents a possible opinion over all binary issues, and two positions are connected if and only if they differ on exactly one issue. According to such a modeling, a candidate can stand on a vertex of the hypercube, which corresponds to communicating the associated political opinion, and attracts voters who agree with her announced position on all issues, but also the voters for whom she is the "closest" candidate, i.e., she announces a position which differs on the least number of issues to their opinion.

Consequently, given a distribution of the voters on the hypercube of issues and the position of her competitors, a candidate may be willing to move strategically from the vertex corresponding to her initial truthful political stance, to another position in the hypercube, in order to obtain a better outcome in the election. We assume that the candidates only have the short term perspective of the election, i.e., they do not seek to maximize the number of votes they receive, but instead aim to be elected or, potentially, to help the election of another candidate that they prefer over the current winner. This game defines a *binary* variant of strategic candidacy that corresponds to a Hotelling-Downs game [10, 11] on a hypercube structure.

In this model, some moves from one position to another in the hypercube of issues may be unlikely to occur, when these positions are too far apart. Indeed, a candidate would not benefit from expressing very contrary opinions because voters would uncover the strategic and insincere aspect of such move, and would not vote for this dishonest candidate. Thus, it seems realistic to assume that only local moves would be performed. This leads to the definition of a new solution concept, called *t-local equilibrium*, which generalizes the notion of Nash equilibrium, and captures stability w.r.t. moves to positions that differ on at most t issues from the current one.

In this article, we investigate the existence of Nash equilibria and of local equilibria in binary strategic candidacy games, both theoretically and empirically, and focus on several natural restrictions, either on the distribution of voters or on the structure of candidates' strategy sets. Specifically, we study the impact of restricting to a single-peaked distribution of voters. Such restriction can be interpreted as a homogeneous voting body in which there exists a *popular* position corresponding to the most frequent political stance; the other positions becoming less and less frequent when moving away from this peak position.[1] Another interesting type of restriction is related to the set of positions in the hypercube a candidate can take. A rationale for this restriction comes from the fact that candidates might not want to deviate too much from their truthful position, representing their real opinions. Moreover, there could be correlations between issues (or more generally some structure over the set of issues) that imply some forbidden positions (e.g., for abortion and against euthanasia).

Most of our results concern the case of two candidates. In general, even a 1-local equilibrium is not guaranteed to exist with only two candidates. Computationally speaking, we show that deciding the existence of a t-local equilibrium is NP-hard for all $t \geq 2$. However, by considering a single-peaked distribution of voters and rather weak additional conditions, we prove that a Nash equilibrium always exists with two candidates. While this restriction allows for the existence of a strongly stable state, this result cannot be extended to more candidates. Indeed, we show that even a 1-local equilibrium may not exist with only three candidates under a uniform distribution of voters, a strong restriction where each position is chosen by exactly the same number of voters. Other positive results can be reached when restricting the candidates' strategies, i.e., when assuming some structure on the possible positions $\mathcal{H}_i$ that each candidate c_i can announce. In particular, when the strategies of one candidate are included in the set of strategies of the other candidate, a 2-local equilibrium is guaranteed to exist with two candidates. Moreover, when the candidates' strategies of the two candidates are restricted to balls of radius one around their preferred position, a 1-local equilibrium always exists. Our existence results for two candidates are summarized in Table 1.

We first report in Section 2 the different directions followed by the related work. Then, we describe our model in Section 3 with the definition of binary strategic candidacy games and their possible restrictions. Our theoretical results regarding the existence of t-local equilibria are then presented in Section 4, with the investigation of restrictions on the distribution of voters on the hypercube of positions, and on the

[1] The idea behind this restriction is also close to *unimodal distributions* where a given preference, called the mode, is more likely to occur than others, and the probability for a preference to occur decreases as one moves away from the mode according to a given distance (see, e.g., [12]).

	Voter distribution				
	General				Single-peaked
	Candidates' strategies				
	General	$\mathcal{H}_1 \cap \mathcal{H}_2 \neq \emptyset$	$\mathcal{H}_2 \subseteq \mathcal{H}_1$	Balls of radius 1	
Nash	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\exists^*$ [Th. 10]
3-local	$\nexists$	$\nexists$ ($K = 3$, $\mathcal{H}_1 = \mathcal{H}_2$) [Prop. 8]		$\nexists$	$\exists^*$
2-local	$\nexists$	$\exists^*$ [Th. 5]	$\exists$ [Th. 19]	$\nexists$ ($K = 3$) [Prop. 23]	$\exists^*$
1-local	$\nexists$ ($K = 3$) [Prop. 4]	$\exists^*$	$\exists$	$\exists$ [Th. 22]	$\exists^*$

Table 1: Summary of our results regarding the existence of Nash equilibria and of t-local equilibria for two candidates. The set of possible strategies for a candidate c_i is denoted by $\mathcal{H}_i$, for $i \in \{1, 2\}$. The cases where an equilibrium is guaranteed to exist are marked with symbol $\exists$ (the symbol $\exists^*$ indicates that we need additional minor conditions for the result to hold), while the cases where an equilibrium may not exist are marked with symbol $\nexists$. The specific conditions under which our negative results hold are specified between parentheses.

candidates' strategies. These theoretical results are then complemented in Section 5 by experiments with simulations run on synthetic data to better understand the practical existence of equilibria, as well as the convergence of game dynamics defined by local deviations. We finally conclude our work by summarizing its key findings and mention several perspectives in Section 6.

2 Related Work

Several attempts to tackle similar problems have been found in the literature, coming from a diversity of areas. The Hotelling-Downs model has existed since its original formulation by Hotelling [10] on the well-known problem of ice-cream vendors positioning themselves strategically on a beach. This idea was later translated to voting theory by Downs [11], adapting the strategic location of vendors to a strategic placement of candidates on a political spectrum. The Hotelling-Downs model (HDM) is one of the most widespread models to interpret scenarios coming both from politics and from economics. A range of variants have been studied over the years, both in the context of *facility location* (the game of companies placing their facilities on a given metric space, trying to attract customers assumed to seek for the closest available seller) [13, 14] and in voting models for positioning of candidates [15].

Sengupta and Sengupta [16] were among the first to make links between the literature of the HDM with that of *strategic candidacy*, an election game where candidates may abstain at will, in order to achieve a result closer to their preference. The original model of *strategic candidacy* was introduced by Dutta et al. [5], being followed along the years by multiple different variants, e.g., mixing strategic voting and strategic candidacy [17], understanding its equilibria [18, 19], or assuming given behaviors for candidates [20, 21].

The first papers (to our knowledge) trying to make the fusion between the Hotelling-Downs model for elections and strategic candidacy are Sabato et al. [6] (with their *real candidacy games*), and Harrenstein et al. [7] (with their *HDM for party* *nominees*). Sabato et al. [6] consider candidates who can strategically choose ideological positions in an interval of the real line to be more attractive to voters. Similarly, our candidates strategically choose positions in a hypercube representing opinions on binary issues, and their set of strategies in the hypercube can be restricted to some reasonable options. However, contrary to the interval strategy sets, in our model, the sets of candidates' strategies are discrete, and thus we could also consider the candidates as political parties who can choose possible party nominees, like in the model by Harrenstein et al. [7].

Quite similar models (although with a different perspective) come from the context of Algorithmic Game Theory, with *Voronoi games*: strategic positioning of players on a metric space, seeking to maximize the number of points that fall the closest to them. Despite the extensive literature on these games for continuous settings and sequential decisions (see, e.g., [22, 23]), the discrete-setting variant of *Voronoi games on graphs* was relatively recently discussed by Dürr and Thang [24] with the complexity analysis of deciding the existence of a Nash equilibrium. In our binary strategic candidacy game, as it is classical in Voronoi games, the voters split their vote among candidates that are the closest to their own opinion's position. However, our candidates do not aim to maximize the number of votes they receive (contrary to *Voronoi games*), but want to get a better outcome for the election (like in *strategic candidacy*). The analysis of our game, which is based on a hypercube, has some similarities with that of Voronoi games in transitive graphs [25], in particular on the importance of *antipodal positions* in the graph.

As mentioned by Harrenstein et al. [7], there is really scarce literature on the HDM for elections with multiple participants and restricted strategy sets (or, somewhat equivalently in their context, multiple parties selecting nominees from a fixed set of possible candidates). One can nevertheless cite a recent follow-up paper to Harrenstein et al. [7] by Deligkas et al. [26], which investigates the parameterized complexity of deciding the existence of a pure Nash equilibrium in HDM for party nominees. Even though similar games have been studied for general graphs, no evidence was found of an attempt to apply the ideas of Hotelling-Downs specifically to a hypercube over issues, as we do in this article. The main idea of such a model comes from the setting of *Judgment Aggregation (JA)* [27]. In this context, Nehring and Puppe [28] have notably defined *general single-peaked structures*, from which we take inspiration to define single-peaked distributions of voters on the hypercube. The use of the Hamming distance in our study was similarly inspired by this field of research (though other alternatives could have been considered from the vast JA literature, see, e.g., [29]).

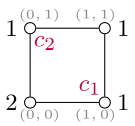
3 The Model

For an integer $k \in \mathbb{N}$, we define $[k] := \{1, \ldots, k\}$. We are given a set of n voters $N = [n]$, and a set of m candidates $C = \{c_1, \ldots, c_m\}$. We assume that the population (voters and candidates) is interested in a fixed number $K \in \mathbb{N}$ of relevant *binary issues* (each of which we denote by $j \in [K]$). All possible opinions on these binary issues are given by the set $\mathcal{H} = \{0, 1\}^K$. A position $p \in \mathcal{H}$ representing a global opinion over all issues is a K-vector $p = (p_1, p_2, \ldots, p_K)$ where $p_j \in \{0, 1\}$ for all $j \in [K]$. The distance between two positions p and p' in $\mathcal{H}$ is defined as the *Hamming distance* between the two corresponding vectors, i.e., $dist(p, p') = |\{j \in [K] : p_j \neq p'_j\}|$. The possible positions can be represented on a *hypercube graph* $G^{\mathcal{H}} := (\mathcal{H}, E)$ where $\{p, p'\} \in E$ iff $dist(p, p') = 1$, for every pair of positions $p, p' \in \mathcal{H}$. The *antipodal position* $\hat{p}$ of position $p \in \mathcal{H}$ is the position where all opinions are reversed compared to p, i.e., $\hat{p}_i = 1 - p_i$ for every $i \in [K]$.

Each voter $v \in N$ and each candidate $c \in C$ is associated with a position on the hypercube, denoted by $p^v \in \mathcal{H}$ and $p^c \in \mathcal{H}$, respectively, representing her opinion about all binary issues. The voters are assumed to focus on the announced opinions of the candidates on the binary issues in order to form their preferences over the candidates. More precisely, the voters prefer the candidates whose announced opinions are closer to theirs. The preferences of each voter $v \in N$ over positions in the hypercube are represented by a weak order $\succsim_v$ over $\mathcal{H}$ such that $p \succsim_v p'$ iff $dist(p, p^v) \leq dist(p', p^v)$ (the strict and symmetric parts of $\succsim_v$ are denoted by $\succ_v$ and $\sim_v$, respectively). Then, the voters can derive, from their fixed preferences over the positions in the hypercube, their preferences over the candidates. The preferences of each voter $v \in N$ over the candidates, w.r.t. a profile of positions $\mathbf{s} = (s_1, \ldots, s_m) \in \mathcal{H}^m$ where s_i is the announced position of candidate $c_i \in C$, can be represented by a weak order $\succsim_v^{\mathbf{s}}$ over C. In short, a voter will prefer candidates that *announce* positions closer to their own opinion's position, i.e., $c_i \succsim_v^{\mathbf{s}} c_j$ iff $s_i \succsim_v s_j$, for every $i, j \in [m]$.

The candidates run for an election whose winner is determined by a *voting rule* $\mathcal{F} : \succsim^{\mathbf{s}} \rightarrow C$, which is a variant of the *plurality rule.* Voting rule $\mathcal{F}$ takes as input the preferences of the voters according to a state $\mathbf{s} \in \mathcal{H}^m$ of announced positions of the candidates, or equivalently, the positions of all voters as well as the description of $\mathbf{s}$, and returns a winning candidate in C. In this variant of plurality, each voter has one point that she divides among the candidates she ranks in the top indifference class of her preference ranking, and $\mathcal{F}$ returns one candidate with the highest score. We make $\mathcal{F}$ resolute by assuming a *deterministic lexicographic tie-breaking rule* based on the linear order $\rhd$ over C such that $c_1 \rhd c_2 \rhd \ldots c_m$. The score of the candidates w.r.t. voting rule $\mathcal{F}$ on preference profile $\succsim^{\mathbf{s}}$ is given by a scoring function $score_{\mathcal{F}}^{\succsim^{\mathbf{s}}} : C \rightarrow \mathbb{R}$ (when the context is clear the parameters may be omitted), which gives the number of points that each candidate gets under our variant of plurality $\mathcal{F}$, and $\mathcal{F}(\succsim^{\mathbf{s}}) \in \arg\max_{c \in C} score_{\mathcal{F}}^{\succsim^{\mathbf{s}}}(c)$.

Example 1 Consider an instance with two issues, two candidates c_1 and c_2, and five voters whose positions are $p^1 = p^2 = (0,0)$, $p^3 = (1,0)$, $p^4 = (0,1)$, and $p^5 = (1,1)$. The two candidates are such that $p^{c_1} = (1,0)$ and $p^{c_2} = (0,1)$ and they announce their truthful position. The voters can be described as weights related to positions in the hypercube as represented below (left), and their preferences over positions and over candidates can be derived as done below (right). Thus, we have $score_{\mathcal{F}}(c_1) = score_{\mathcal{F}}(c_2) = 2.5$ and c_1 wins by the tie-breaking rule.



$$\begin{array}{rccccccccccc}
1,2: & (0,0) & \succ & (1,0) & \sim & (0,1) & \succ & (1,1) & \Rightarrow & c_1 & \sim & c_2 \\
3: & (1,0) & \succ & (0,0) & \sim & (1,1) & \succ & (0,1) & \Rightarrow & c_1 & \succ & c_2 \\
4: & (0,1) & \succ & (0,0) & \sim & (1,1) & \succ & (1,0) & \Rightarrow & c_2 & \succ & c_1 \\
5: & (1,1) & \succ & (1,0) & \sim & (0,1) & \succ & (0,0) & \Rightarrow & c_1 & \sim & c_2
\end{array}$$

3.1 The Binary Strategic Candidacy (BSC) Game

The candidates may announce opinions on the issues that do not exactly fit their truthful opinion, in order to alter the outcome of the election towards one they consider *better*. Therefore, analogously to the voters, each candidate $c \in C$ also expresses preferences over the candidates, that are represented by a weak order $\succsim_c$ over C. Basically, since they run for the election, all candidates prefer to be elected than that another candidate is elected, i.e., for every candidate $c \in C$, $\succsim_c$ is such that $c \succ_c c'$ for every $c' \in C \setminus \{c\}$. Note that the candidates may not be willing to announce any possible position in the hypercube (they may not want to lie too much compared to their truthful position). The subset of possible announced positions for candidate $c_i \in C$ is given by $\mathcal{H}_i \subseteq \mathcal{H}$ where $p^{c_i} \in \mathcal{H}_i$.

How the candidates can strategize by advertising political views can be modeled by a *strategic game*:[2] the *Binary Strategic Candidacy (BSC) game*. In this game, the set of players corresponds to the set of candidates, the set of strategies of each candidate $c_i \in C$ is given by $\mathcal{H}_i$, and a strategy profile $\mathbf{s}$ is the tuple of announced positions $\mathbf{s} = (s_1, \ldots, s_m)$ where $s_i \in \mathcal{H}_i$ for each candidate $c_i \in C$. We will classically refer to a strategy profile $\mathbf{s}$ as a *state* of the game and, by abuse of notation, we will directly write $\mathcal{F}(\mathbf{s})$ to denote the winner of the election at state $\mathbf{s}$ according to the fixed preferences of the voters over the positions. A state $\mathbf{s}$ is only evaluated via its winner $\mathcal{F}(\mathbf{s})$; namely, a player will prefer one state over another if she prefers the winner of the election that is obtained in such state. Notice that a candidate's satisfaction is not determined by the number of votes she receives, but only by the *final outcome* of the election. It follows that a candidate will seek to increase the number of votes she receives only if this either makes her the new winner or, if not possible, achieves a winner that she prefers over the current one. In other words, candidate c_i has a better response from state $\mathbf{s}$ if there exists a position $s'_i \in \mathcal{H}_i$ such that $\mathcal{F}((s'_i, s_{-i})) \succ_{c_i} \mathcal{F}(\mathbf{s})$. We can thus redefine the well-known solution concept of *Nash equilibrium* for the BSC game.

Definition 1 (Nash equilibrium) A state $\mathbf{s} \in \prod_{i=1}^{m} \mathcal{H}_i$ is a Nash equilibrium if there is no strategy $s'_i \in \mathcal{H}_i$ for a candidate $c_i \in C$ such that $\mathcal{F}((s'_i, s_{-i})) \succ_{c_i} \mathcal{F}(\mathbf{s})$.

A Nash equilibrium is immune to unilateral deviations of candidates to another position that would strictly improve the outcome of the election with respect to their preferences. The considered deviations for a candidate c_i can be of any type within $\mathcal{H}_i$. However, it may not be realistic for a candidate to pass from one announced position to a radically different one: the voters may not trust her. We thus relax the solution concept of Nash equilibrium by considering stability w.r.t. reasonable deviations that are not too far away from the candidate's current position. This solution concept is the *t-local equilibrium*, given a maximum distance $t \in [K]$.

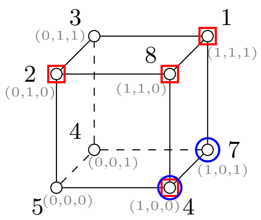
Definition 2 (t-local equilibrium) A state $\mathbf{s} \in \prod_{i=1}^{m} \mathcal{H}_i$ is a t-local equilibrium if there is no strategy $s'_i \in \mathcal{H}_i$ for a candidate $c_i \in C$ such that $dist(s'_i, s_i) \leq t$ and $\mathcal{F}((s'_i, s_{-i})) \succ_{c_i} \mathcal{F}(\mathbf{s})$.

[2] The interested reader shall see, e.g., Nisan et al. [4] for a standard reference in Game Theory.

By definition, a t-local equilibrium is a t'-local equilibrium for every $1 \leq t' \leq t \leq K$.

Beyond the local constraint, the candidates' deviations are also constrained by the shape of their set of strategies. In particular, a candidate cannot perform a local deviation whose distance exceeds the maximum distance between two positions in her strategy set. Hence, a Nash equilibrium is equivalent to a t-local equilibrium where $t \geq \max_{i \in [m]} \max_{p,p' \in \mathcal{H}_i} dist(p, p')$, and in general is equivalent to a K-local equilibrium. Therefore, in a given BSC game, if a Nash equilibrium exists, then a t-local equilibrium exists for every $t \in [K]$, and if a t-local equilibrium does not exist, then no Nash equilibrium can exist. We illustrate in the next example the solution concept of t-local equilibrium, which is a relaxation of the concept of Nash equilibrium.

Example 2 Consider a BSC game with $m = 2$ candidates, $n = 34$ voters, and $K = 3$ issues. The sets of strategies for candidates c_1 and c_2 are $\mathcal{H}_1 := \{(0, 1, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ and $\mathcal{H}_2 = \{(1, 0, 0), (1, 0, 1)\}$, respectively. The distribution of voters on the hypercube as well as the candidates' strategies are represented below on the left (red squares for $\mathcal{H}_1$ and blue circles for $\mathcal{H}_2$). The table below (right) reports all possible states of the game; the number of votes that each candidate gets is given for each state, and it is written in bold to represent the winner. The Nash deviations are denoted by an arrow towards a best response for the candidate mentioned next to the arrow.



	$s_2 \in \mathcal{H}_2$: $(1,0,0)$	$s_2 \in \mathcal{H}_2$: $(1,0,1)$
$s_1 \in \mathcal{H}_1$: $(0,1,0)$	$(14, \mathbf{20})$	$(\mathbf{18}, 16)$
$(1,0,0)$	$(\mathbf{17}, 17)$	$(\mathbf{19}, 15)$
$(1,1,0)$	$(14, \mathbf{20})$	$(16.5, \mathbf{17.5})$
$(1,1,1)$	$(14.5, \mathbf{19.5})$	$(14, \mathbf{20})$

In this BSC game, there are exactly two Nash equilibria, namely $((1, 0, 0), (1, 0, 0))$ and $((1, 0, 0), (1, 0, 1))$, which are marked in red in the previous table. Since the maximal distance between two positions in a candidate's strategy set is equal to two, these two states are also the only 2-local equilibria. Since no Nash deviation is possible from these two states, no 1-local deviation is possible either, therefore they are also 1-local equilibria. This BSC game also contains three other 1-local equilibria: $((1, 1, 1), (1, 0, 0))$, $((0, 1, 0), (1, 0, 0))$, and $((1, 1, 1), (1, 0, 1))$. These are, however, *not* 2-local equilibria, since they allow for improving deviations of distance two.

3.2 Restrictions on the BSC Game

Distribution of voters

Each voter $v \in N$ is characterized by her position $p^v \in \mathcal{H}$. This means that we can alternatively formulate the set of voters as a *distribution of voters over* $\mathcal{H}$, i.e., a function $f_N : \mathcal{H} \rightarrow \mathbb{N}$ such that $\sum_{p \in \mathcal{H}} f_N(p) = n$, counting how many voters have each position $p \in \mathcal{H}$ as their own opinion's position. By abuse of notation, for $S \subseteq \mathcal{H}$, we denote by $f_N(S)$ the number of voters whose positions lie in S, i.e., $f_N(S) := \sum_{p \in S} f_N(p)$.

Let $[x, z] := \{y \in \mathcal{H}$: there is a shortest path in $G^{\mathcal{H}}$ between x and z passing through $y\}$ denote all positions *between* x *and* z, for every $x, z \in \mathcal{H}$. A distribution f_N is said to be *single-peaked* if there exists a peak position $p^* \in \mathcal{H}$ such that for every pair of positions $x, y \in \mathcal{H}$, $y \in [x, p^*]$ implies $f_N(x) \leq f_N(y)$. We will also say that f_N is single-peaked with respect to p^*, and we will call p^* a *peak* of the distribution. This definition encodes the idea of having a most popular opinion p^* such that, when walking away from it, we find only positions that are at most as popular. In a way, the *peak* describes some sort of *attractive popular opinion* within the population. A particular case of a single-peaked distribution is the *uniform distribution*, in which $f_N : \mathcal{H} \to \mathbb{N}$ is constant.[3]

Candidates' preferences

Beyond the fact that the preferences of the candidates are such that each candidate strictly prefers herself to any other candidate, they can be of several types. We will particularly focus in the article on the following types:

- *fixed*: the candidates' preferences are not affected by the position chosen by the other candidates;[4]
- *narcissistic*: the candidates do not care about the winner if they are not elected, i.e., for every candidate $c \in C$, $\succsim_c$ is such that $c' \sim_c c''$ for every $c', c'' \in C \setminus \{c\}$.

Note that the two types of candidates' preferences coincide when there are only two candidates, and that narcissistic preferences are a specific type of fixed preferences. It follows that a t-local equilibrium under fixed candidates' preferences is also a t-local equilibrium under narcissistic candidates' preferences.

Candidates' strategies

It would seem unnatural if the only possible positions that a candidate may announce were, e.g., antipodal positions. Therefore, a realistic assumption on the set of strategies of a candidate is its connectedness in the hypercube. Another natural restriction would be to assume that the set of strategies of candidate $c_i \in C$ is a *ball of a given radius* b, meaning that all positions at distance at most b from her truthful position are positions that she accepts to announce (a candidate accepts to lie on at most b issues no matter which they are), i.e., $\mathcal{H}_i := \{p \in \mathcal{H} : dist(p, p^{c_i}) \leq b\}$.[5] Notice that, despite the set $\mathcal{H}_i$ being possibly arbitrarily constrained, the notion of t-local equilibrium still does not necessarily coincide with that of a Nash equilibrium (e.g., even with $\mathcal{H}_i$ being a *ball of radius* 1, 2-local deviations are still possible).

Case of $m = 2$ *candidates*

One can exploit the geometric structure of the hypercube, which provides particular insights for the case of two candidates. When we deal with two candidates,

[3] This distribution could be seen as the deterministic version of impartial culture where all possible preferences or opinions are equally likely.

[4] Note that candidates' preferences determined by the distances between their truthful and their rivals' truthful positions, are a particular case of *fixed preferences*. In that case, due to the narcissistic behavior of candidates, we nevertheless assume that each candidate still prefers herself to any other candidate, even if they share the same truthful position.

[5] Note that candidates' strategies that are balls induce a symmetric neighborhood around the truthful position, which implicitly assumes independence of the issues.

the hypercube $\mathcal{H}$ can be easily partitioned into *sets of influence* associated with each candidate and a *set of indifferent positions*. For an index $i \in \{1,2\}$, let c_{-i} denote candidate c_{3-i}. Given a strategy profile $\mathbf{s} = (s_1, s_2)$, the set of influence of candidate c_i for $i \in \{1,2\}$ is denoted by $P_i^{\mathbf{s}}$ and represents the set of positions which are strictly closer to the announced position of c_i than to the one of c_{-i}, i.e., $P_i^{\mathbf{s}} := \{p \in \mathcal{H} : dist(p, s_i) < dist(p, s_{-i})\}$. Given a strategy profile $\mathbf{s} = (s_1, s_2)$, the set of indifferent positions is defined by $I^{\mathbf{s}} := \{p \in \mathcal{H} : dist(p, s_i) = dist(p, s_{-i})\}$. It follows that, given a strategy profile $\mathbf{s} = (s_1, s_2)$, the set of all possible positions can be partitioned as follows: $\mathcal{H} = P_1^{\mathbf{s}} \uplus P_2^{\mathbf{s}} \uplus I^{\mathbf{s}}$ (where $\uplus$ denotes the disjoint set union). This means that for every voter $v \in N$, it holds that $p^v \in P_i^{\mathbf{s}} \Leftrightarrow c_i \succ_v^{\mathbf{s}} c_{-i}$ and $p^v \in I^{\mathbf{s}} \Leftrightarrow c_i \sim_v^{\mathbf{s}} c_{-i}$. In other words, $P_i^{\mathbf{s}}$ contains all the positions for which the voters are guaranteed to strictly prefer c_i over c_{-i} under strategy profile $\mathbf{s}$, whereas $I^{\mathbf{s}}$ contains all those for which the voters are indifferent between the two candidates. Note that the voters whose opinion's position lies in $I^{\mathbf{s}}$ do not matter for the computation of the scores of the two candidates, since their vote is equally divided between the two candidates. Therefore, the winner w.r.t. $\mathcal{F}$ in state $\mathbf{s}$ only depends on the number of voters positioned in both $P_1^{\mathbf{s}}$ and $P_2^{\mathbf{s}}$, i.e., $\mathcal{F}(\mathbf{s}) \in \arg\max_{c_i \in C} f_N(P_i^{\mathbf{s}})$. Hence, understanding the structure of the sets of influence is key for the analysis of the game.

First observe that we can focus on the parts of the strategy positions that are different between the two candidates. Given $\mathbf{s} = (s_1, s_2)$, let $X_{=}^{\mathbf{s}}$ and $X_{\neq}^{\mathbf{s}}$ denote the sets of issues on which positions s_1 and s_2 agree and disagree, respectively, i.e., $X_{=}^{\mathbf{s}} := \{j \in [K] : (s_1)_j = (s_2)_j\}$ and $X_{\neq}^{\mathbf{s}} := \{j \in [K] : (s_1)_j \neq (s_2)_j\}$. By definition, we have $[K] = X_{=}^{s} \uplus X_{\neq}^{s}$ and $|X_{\neq}^{\mathbf{s}}| = dist(s_1, s_2)$. Let $dist_{\neq}^{\mathbf{s}}(\cdot,\cdot)$ denote the distance calculated only on the issues of $X_{\neq}^{\mathbf{s}}$. The sets of influence can be defined only based on $dist_{\neq}^{\mathbf{s}}(\cdot,\cdot)$.

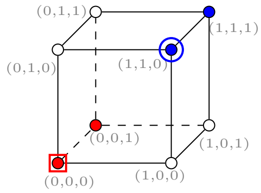
Observation 3.1 When $m = 2$, for every state $\mathbf{s} \in \mathcal{H}_1 \times \mathcal{H}_2$, $i \in \{1,2\}$, and position $p \in \mathcal{H}$, we have $p \in P_i^{\mathbf{s}} \Leftrightarrow dist_{\neq}^{\mathbf{s}}(p, s_i) < dist_{\neq}^{\mathbf{s}}(p, s_{-i})$, and $p \in I^{\mathbf{s}} \Leftrightarrow dist_{\neq}^{\mathbf{s}}(p, s_i) = dist_{\neq}^{\mathbf{s}}(p, s_{-i})$.

Secondly, we can observe that the sets of influence can be defined w.r.t. the distance between the strategy positions of the two candidates. Given $\mathbf{s} = (s_1, s_2)$ and $r^{\mathbf{s}} := dist(s_1, s_2)$, let $d_{\mathbf{s}}$ denote the *critical distance* up to which any given candidate has ensured influence, i.e., $d_{\mathbf{s}} := \lceil \frac{r^{\mathbf{s}}}{2} \rceil - 1$.

Observation 3.2 When $m = 2$, for every state $\mathbf{s} \in \mathcal{H}_1 \times \mathcal{H}_2$, $i \in \{1,2\}$, and position $p \in \mathcal{H}$, we have $p \in P_i^{\mathbf{s}} \Leftrightarrow dist_{\neq}^{\mathbf{s}}(p, s_i) \leq d_{\mathbf{s}}$ and $p \in I^{\mathbf{s}} \Leftrightarrow r^{\mathbf{s}}$ is even and $dist_{\neq}^{\mathbf{s}}(p, s_i) = \frac{r^{\mathbf{s}}}{2}$.

Thus, $r^{\mathbf{s}}$ is even iff $I^{\mathbf{s}} \neq \emptyset$. We illustrate the previous remarks in the next example.

Example 3 Consider a BSC game with $m = 2$ candidates, and $K = 3$ issues. We illustrate below a state $\mathbf{s} = (s_1, s_2)$ where strategy $s_1 = (0,0,0)$ is represented by a red square and $s_2 = (1,1,0)$ is represented by a blue circle in the hypercube. The positions in the set of influence of candidate c_1 (resp., c_2) are marked in red (resp., blue).



The set of issues on which s_1 and s_2 agree is $X^{\mathbf{s}}_{=} = \{3\}$ and on which they disagree is $X^{\mathbf{s}}_{\neq} = \{1, 2\}$. The distance between their positions is equal to $r^{\mathbf{s}} = |X^{\mathbf{s}}_{\neq}| = 2$, and the critical distance is equal to $d_{\mathbf{s}} = 0$. The sets of influence of each candidate in $\mathbf{s}$ are $P^{\mathbf{s}}_1 = \{(0, 0, 0), (0, 0, 1)\}$ and $P^{\mathbf{s}}_2 = \{(1, 1, 0), (1, 1, 1)\}$, and the indifference set is $I^{\mathbf{s}} = \{(0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1)\}$. As remarked in Observation 3.2, in this example, $p \in P^{\mathbf{s}}_i$ for $i \in \{1, 2\}$ when $dist^{\mathbf{s}}_{\neq}(p, s_i) \leq d_{\mathbf{s}} = 0$ and $p \in I^{\mathbf{s}}$ when $dist^{\mathbf{s}}_{\neq}(p, s_1) = dist^{\mathbf{s}}_{\neq}(p, s_2) = 1$.

An interesting further remark is that when we change one of the two strategy positions of a state on exactly one issue, then no position can directly pass from an influence set to another, it must intermediately pass by the indifference set, as stated in the next lemma. We denote by $L^{\mathbf{s}}_i$ the set of positions in $P^{\mathbf{s}}_i$ that are at the limit of the set of influence of candidate c_i in $\mathbf{s}$, i.e., $L^{\mathbf{s}}_i := \{p \in P^{\mathbf{s}}_i : dist(p, s_i) = dist(p, s_{-i}) - 1\}$. For a given subset $P \subseteq \mathcal{H}$, let $P_{[x=e]}$ denote the subset of positions from P whose value on issue x is equal to $e \in \{0, 1\}$, i.e., $P_{[x=e]} := \{p \in P : p_x = e\}$.

Lemma 1 *When $m = 2$, if candidate c_i for $i \in \{1, 2\}$ performs a 1-local deviation from state $\mathbf{s} = (s_i, s_{-i})$ to state $\mathbf{s}' = (s'_i, s_{-i})$ where position strategies s_i and s'_i differ only on issue $x \in [K]$, then:*

- *if $r^{\mathbf{s}}$ is odd, then $r^{\mathbf{s}'}$ is even and the positions which were at the limit of the set of influence of candidate c_i (resp., c_{-i}) and which do not share (resp., do share) the same value as the new strategy s'_i on issue x move to the indifference set of the new state, i.e., $P^{\mathbf{s}'}_i = P^{\mathbf{s}}_i \setminus (L^{\mathbf{s}}_{i\ [x=1-(s'_i)_x]})$, $P^{\mathbf{s}'}_{-i} = P^{\mathbf{s}}_{-i} \setminus (L^{\mathbf{s}}_{-i\ [x=(s'_i)_x]})$ and $I^{\mathbf{s}'} = (L^{\mathbf{s}}_{i\ [x=1-(s'_i)_x]}) \cup (L^{\mathbf{s}}_{-i\ [x=(s'_i)_x]})$,*
- *if $r^{\mathbf{s}}$ is even, then $r^{\mathbf{s}'}$ is odd and the set of influence of candidate c_i (resp., c_{-i}) is augmented by the positions in the previous indifference set which have (resp., do not have) the same value as the new strategy s'_i on issue x, i.e., $P^{\mathbf{s}'}_i = P^{\mathbf{s}}_i \cup I^{\mathbf{s}}_{[x=(s'_i)_x]}$, $P^{\mathbf{s}'}_{-i} = P^{\mathbf{s}}_{-i} \cup I^{\mathbf{s}}_{[x=1-(s'_i)_x]}$, and $I^{\mathbf{s}'} = \emptyset$.*

Proof We have $r^{\mathbf{s}} = dist(s_i, s_{-i})$ and $dist(s_i, s'_i) = 1$, therefore $|r^{\mathbf{s}} - r^{\mathbf{s}'}| = 1$ and $r^{\mathbf{s}'}$ has a different parity from $r^{\mathbf{s}}$.

Since $|d_{\mathbf{s}} - d_{\mathbf{s}'}| \leq 1$, given a position $p \in P^{\mathbf{s}}_i$ (resp., $p \in P^{\mathbf{s}}_{-i}$), if $dist(p, s_i) < d_{\mathbf{s}}$ (resp., $dist(p, s_{-i}) < d_{\mathbf{s}}$) then $dist(p, s'_i) \leq d_{\mathbf{s}'}$ (resp., $dist(p, s_{-i}) \leq d_{\mathbf{s}'}$), implying that still $p \in P^{\mathbf{s}'}_i$ (resp., $p \in P^{\mathbf{s}'}_{-i}$).

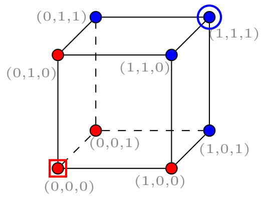
- Suppose that $r^{\mathbf{s}}$ is even. For a given position $p \in P^{\mathbf{s}}_i$ (resp., $p \in P^{\mathbf{s}}_{-i}$) such that $dist(p, s_i) = d_{\mathbf{s}}$ (resp., $dist(p, s_{-i}) = d_{\mathbf{s}}$) we have $dist(p, s_i) = dist(p, s_{-i}) - 2$ (resp., $dist(p, s_{-i}) = dist(p, s_i) - 2$). Therefore, because $dist(s_i, s'_i) = 1$, we get that $dist(p, s'_i) <$

$dist(p, s_{-i})$ (resp., $dist(p, s_{-i}) < dist(p, s_i'))$ and thus still $p \in P_i^{\mathbf{s}'}$ (resp., $p \in P_{-i}^{\mathbf{s}'}$). Hence, $P_j^{\mathbf{s}} \subseteq P_j^{\mathbf{s}'}$ for every $j \in \{1, 2\}$.

By definition, for every position $p \in I^{\mathbf{s}}$, we have $dist(p, s_i) = dist(p, s_{-i})$. Because $dist(s_i, s_i') = 1$, if p and s_i' agree on issue x, then we have $dist(p, s_i') = dist(p, s_i) - 1 < dist(p, s_{-i})$, and thus $p \in P_i^{\mathbf{s}'}$. Otherwise, i.e., if p and s_i' differ on issue x, then we have $dist(p, s_i') = dist(p, s_i) + 1 > dist(p, s_{-i})$, and thus $p \in P_{-i}^{\mathbf{s}'}$.

- Suppose that $r^{\mathbf{s}}$ is odd. Therefore, we have that $I^{\mathbf{s}} = \emptyset$. For a given position $p \in P_i^{\mathbf{s}}$ (resp., $p \in P_{-i}^{\mathbf{s}}$) such that $dist(p, s_i) = d_{\mathbf{s}}$ (resp., $dist(p, s_{-i}) = d_{\mathbf{s}}$), we have $dist(p, s_i) = dist(p, s_{-i}) - 1$ (resp., $dist(p, s_{-i}) = dist(p, s_i) - 1$). Therefore, because $dist(s_i, s_i') = 1$, if p and s_i' agree on issue x (resp., p and s_i' differ on issue x), then we have $dist(p, s_i') = dist(p, s_i) - 1 = dist(p, s_{-i}) - 2$ (resp., $dist(p, s_i') = dist(p, s_i) + 1 = dist(p, s_{-i}) + 2$), and thus still $p \in P_i^{\mathbf{s}'}$ (resp., $p \in P_{-i}^{\mathbf{s}'}$). Otherwise, i.e., if p and s_i' differ on issue x (resp., p and s_i' agree on issue x), then we have $dist(p, s_i') = dist(p, s_i) + 1 = dist(p, s_{-i})$ (resp., $dist(p, s_i') = dist(p, s_i) - 1 = dist(p, s_{-i})$), and thus $p \in I^{\mathbf{s}'}$. $\square$

Example 3 (continued) For the state $\mathbf{s} = (s_1, s_2)$ with $s_1 = (0, 0, 0)$ and $s_2 = (1, 1, 0)$, consider a 1-local deviation by candidate c_2 who changes her announced position only on issue 3, leading to the new strategy $s_2' = (1, 1, 1)$, and thus to the new state $\mathbf{s}' = (s_1, s_2')$. We illustrate below the state $\mathbf{s}'$ where $s_1 = (0, 0, 0)$ is represented by a red square and $s_2' = (1, 1, 1)$ is represented by a blue circle in the hypercube. The positions in the new set of influence of candidate c_1 (resp., c_2) are marked in red (resp., blue).



Now the distance between the candidates' positions is equal to $r^{\mathbf{s}'} = 3$, and the critical distance is equal to $d_{\mathbf{s}} = 1$. Since $r^{\mathbf{s}}$ is even, we now have $r^{\mathbf{s}'}$ odd and an empty indifference set in $\mathbf{s}'$, i.e., $I^{\mathbf{s}'} = \emptyset$. Following Lemma 1, the set of influence of candidate c_1 (resp., c_2) from $\mathbf{s}$ to $\mathbf{s}'$ has been augmented by the positions in the indifference set of $\mathbf{s}$ which do not have (resp., do have) the same value as the new strategy of c_2 on the issue of deviation, i.e., the positions which have value 0 on issue 3 (resp., value 1 on issue 3). More precisely, we have $P_1^{\mathbf{s}'} = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0)\}$ and $P_2^{\mathbf{s}'} = \{(1, 1, 0), (1, 1, 1), (0, 1, 1), (1, 0, 1)\}$.

Finally, one can observe that the set of influence of a candidate is composed of the antipodal positions of the positions in the set of influence of the other candidate, i.e., for every state $\mathbf{s} \in \mathcal{H}_1 \times \mathcal{H}_2$, $i \in \{1, 2\}$, and position $p \in \mathcal{H}$, we have $p \in P_i^{\mathbf{s}}$ iff $\hat{p} \in P_{-i}^{\mathbf{s}}$, and $p \in I^{\mathbf{s}}$ iff $\hat{p} \in I^{\mathbf{s}}$.

Lemma 2 *When $m = 2$, for every state $\mathbf{s} \in \mathcal{H}_1 \times \mathcal{H}_2$, $i \in \{1, 2\}$, and position $p \in \mathcal{H}$, we have $p \in P_i^{\mathbf{s}} \Leftrightarrow \hat{p} \in P_{-i}^{\mathbf{s}}$, and $p \in I^{\mathbf{s}} \Leftrightarrow \hat{p} \in I^{\mathbf{s}}$.*

Proof By definition of antipodal positions, we have that $dist(p,x) = K - dist(\hat{p},x)$ for every pair of positions $p, x \in \mathcal{H}$. Therefore, for $i \in \{1,2\}$, if $p \in P_i^{\mathbf{s}}$ for a given state $\mathbf{s} \in \mathcal{H}_1 \times \mathcal{H}_2$, we have by definition $dist(p,s_i) < dist(p,s_{-i})$. It follows that $K - dist(\hat{p},s_i) < K - dist(\hat{p},s_{-i})$, and thus $dist(\hat{p},s_{-i}) < dist(\hat{p},s_i)$ and hence, by definition, $\hat{p} \in P_{-i}^{\mathbf{s}}$. The equivalence follows from the fact that $\hat{\hat{p}} = p$. For a position $p \in \mathcal{H}$ and a given state $\mathbf{s} \in \mathcal{H}_1 \times \mathcal{H}_2$, if $p \in I^{\mathbf{s}}$ then, by definition, we have $dist(p,s_i) = dist(p,s_{-i})$, which implies $K - dist(\hat{p},s_i) = K - dist(\hat{p},s_{-i})$, and thus $dist(\hat{p},s_i) = dist(\hat{p},s_{-i})$. Hence, $\hat{p} \in I^{\mathbf{s}}$. □

Example 3 (continued) In the previous example, we can indeed observe that, for both states $\mathbf{s}$ and $\mathbf{s}'$, for each position belonging to a set of influence its antipodal position belongs to the other set of influence.

Consequently, the sets of influence always have the same size for both candidates. Hence, under a uniform distribution of voters, both candidates get the same score in all states, ensuring the existence of Nash equilibria.

Proposition 3 *When $m = 2$, every state of a BSC game is a Nash equilibrium under a uniform distribution of voters.*

Proof By Lemma 2, for every state $\mathbf{s} \in \mathcal{H}_1 \times \mathcal{H}_2$, we have $|P_1^{\mathbf{s}}| = |P_2^{\mathbf{s}}|$. If we have a uniform distribution of voters, then each position represents the opinion of exactly $w := n/2^K$ voters. Therefore, $score(c_1) = score(c_2) = w|P_1^{\mathbf{s}}| + \frac{w}{2}|I^{\mathbf{s}}|$, and thus no change in strategy will ever improve the outcome for any of the players. □

As we will see in the upcoming section, Proposition 3 will fail to hold for more general voter distributions, where Nash equilibria are generally quite rare.

4 On the Existence of a Local Equilibrium

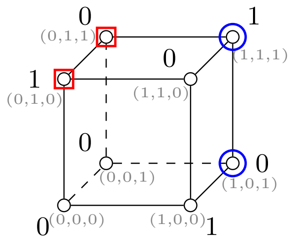
In this section we provide results regarding the existence (or not) of local equilibria in the BSC game under different scenarios. As previously stated, most of our results are focused on the case of $m = 2$ candidates, and the existence is usually guaranteed under additional conditions either on the voter distribution or the candidates' strategy sets.

4.1 Unrestricted Setting

First, a Nash equilibrium may fail to exist (in general) in the BSC game, since even a 1-local equilibrium may not exist even in rather restricted classes of games.

Proposition 4 *A 1-local equilibrium may not exist in a BSC game even when $m = 2$, and $K = 3$.*

Proof Consider a BSC game with $m = 2$ candidates, $n = 3$ voters and $K = 3$ issues. The sets of candidates' strategies are $\mathcal{H}_1 := \{(0,1,0),(0,1,1)\}$ and $\mathcal{H}_2 := \{(1,0,1),(1,1,1)\}$. The distribution of voters on the hypercube as well as the candidates' strategies are represented below on the left (red squares for $\mathcal{H}_1$ and blue circles for $\mathcal{H}_2$). The table below (right) reports all possible states of the game; the number of votes that each candidate gets is given for each state, and it is written in bold to represent the winner. From each of these states, there is a 1-local deviation, denoted by an arrow towards a best response for the candidate mentioned next to the arrow.



		$s_2 \in \mathcal{H}_2$		
		$(1,0,1)$		$(1,1,1)$
$s_1 \in \mathcal{H}_1$	$(0,1,0)$	$(1,\mathbf{2})$	$\xleftarrow{c_2}$	$(\mathbf{1.5},1.5)$
		$c_1 \downarrow$		$\uparrow c_1$
	$(0,1,1)$	$(\mathbf{1.5},1.5)$	$\xrightarrow{c_2}$	$(1,\mathbf{2})$

□

The counterexample presented in Proposition 4 can be easily generalized to higher values of K and n. Indeed, since we do not have any restriction on the voter distribution, the example can be easily *embedded* in a hypercube of dimension $K \geq 4$. Namely, it suffices to consider the same example spanning only a sub-hypercube of dimension 3 (say, using only the first three coordinates), and to take the rest of the positions to have 0 voters. Similarly, the same example can be posed for arbitrarily large values of n (e.g., it is directly generalizable for multiples of 3). Regarding the number of candidates m, however, it is unclear how this particular example might be extended to more than two candidates (though some different counterexamples do exist, as we shall see in Proposition 18). We shall notice that all the counterexamples presented in this section (namely, Propositions 6 and 8) can be similarly generalized to larger K and n (and, analogously, for larger m the direct extension of the provided examples is not as straightforward).

Despite the negative result from Proposition 4, a 2-local equilibrium can be guaranteed to exist in our game if we allow the two candidates to enter a specific set of *majoritarian* positions on the hypercube. Namely, consider position $p^{maj} \in \mathcal{H}$, given by the *majority rule* from Judgment Aggregation [27], in which the *majoritarian view* (according to the voter distribution) is chosen on each individual issue. That is, take p^{maj} to be such that for all $j \in [K]$, $(p^{maj})_j \in \arg\max_{e \in \{0,1\}} f_N(\mathcal{H}_{[j=e]})$. Notice that, whenever n is even, we may have the set of voters perfectly split between both opinions. In order to consider our statement in its most general form, we define as $\mathcal{H}^{maj}$ the set of majoritarian positions, i.e., $\mathcal{H}^{maj} := \{p \in \mathcal{H} \;:\; \forall j \in [K],\ p_j \in \arg\max_{e \in \{0,1\}} f_N(\mathcal{H}_{[j=e]})\}$. We show below that the state where each of the two candidates announces the same majoritarian position is a 2-local equilibrium.

Theorem 5 *There always exists a 2-local equilibrium in a BSC game when $m = 2$, and $\mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}^{maj} \neq \emptyset$.*

Proof Let p^{maj} be an element of $\mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}^{maj}$. Consider state $\mathbf{s}^0 = (p^{maj}, p^{maj}) \in \mathcal{H}_1 \times \mathcal{H}_2$. By the tie-breaking rule, c_1 wins in this state. Any 1-local deviation that c_2 might perform corresponds to moving from position p^{maj} on exactly one issue $x \in [K]$, leading to state $\mathbf{s}^x$. By Lemma 1, we have $P_2^{\mathbf{s}^x} = \mathcal{H}_{[x=1-(p^{maj})_x]}$ and $P_1^{\mathbf{s}^x} = \mathcal{H}_{[x=(p^{maj})_x]}$. By definition of $p^{maj} \in \mathcal{H}^{maj}$, $\mathcal{H}_{[x=(p^{maj})_x]}$ is the *half* of $\mathcal{H}$ with most voters (not necessarily strictly, as n might be even), i.e., $f_N(\mathcal{H}_{[x=(p^{maj})_x]}) \geq f_N(\mathcal{H}_{[x=1-(p^{maj})_x]})$. Thus, we directly have $f_N(P_1^{\mathbf{s}^x}) \geq f_N(P_2^{\mathbf{s}^x})$, and c_1 still wins the election in $\mathbf{s}^x$. Therefore, this 1-local deviation is not improving for c_2. Hence, $\mathbf{s}^0$ is a 1-local equilibrium.

Consider now any 2-local deviation for c_2 from $\mathbf{s}^0$, which corresponds to shifting the strategy of c_2 on two issues, say $x, y \in [K]$. This 2-local deviation results in state $s^{xy} := (p^{maj}, s_2^{xy})$ where s_2^{xy} is the same as p^{maj} except on the two issues x and y. The move from $\mathbf{s}^0$ to s^{xy} can be decomposed into two 1-local deviations of c_2 from $\mathbf{s}^0$ to $\mathbf{s}^x$ and then from $\mathbf{s}^x$ to $\mathbf{s}^{xy}$ (or equivalently with $\mathbf{s}^y$ as an intermediary step). By Lemma 1 on these two steps, we thus obtain that $P_1^{\mathbf{s}^{xy}} = \mathcal{H}_{[x=(p^{maj})_x \wedge y=(p^{maj})_y]}$and $P_2^{\mathbf{s}^{xy}} = \mathcal{H}_{[x=1-(p^{maj})_x \wedge y=1-(p^{maj})_y]}$. Again, by definition of $p^{maj} \in \mathcal{H}^{maj}$, we have that $f_N(\mathcal{H}_{[z=(p^{maj})_z]}) \geq f_N(\mathcal{H}_{[z=1-(p^{maj})_z]})$ for every issue $z \in [K]$. Moreover, by decomposition of sets, we have $f_N(\mathcal{H}_{[z=(p^{maj})_z]}) = f_N(\mathcal{H}_{[z=(p^{maj})_z \wedge z'=(p^{maj})_{z'}]}) + f_N(\mathcal{H}_{[z=(p^{maj})_z \wedge z'=1-(p^{maj})_{z'}]})$ for every pair of issues $z, z' \in [K]$. It follows that:

$$\begin{aligned}
& f_N(\mathcal{H}_{[x=(p^{maj})_x \wedge y=(p^{maj})_y]}) + f_N(\mathcal{H}_{[x=(p^{maj})_x \wedge y=1-(p^{maj})_y]}) \\
& \qquad \geq f_N(\mathcal{H}_{[x=1-(p^{maj})_x \wedge y=(p^{maj})_y]}) + f_N(\mathcal{H}_{[x=1-(p^{maj})_x \wedge y=1-(p^{maj})_y]}) \\
\text{and } & f_N(\mathcal{H}_{[y=(p^{maj})_y \wedge x=(p^{maj})_x]}) + f_N(\mathcal{H}_{[y=(p^{maj})_y \wedge x=1-(p^{maj})_x]}) \\
& \qquad \geq f_N(\mathcal{H}_{[y=1-(p^{maj})_y \wedge x=(p^{maj})_x]}) + f_N(\mathcal{H}_{[y=1-(p^{maj})_y \wedge x=1-(p^{maj})_x]}).
\end{aligned}$$

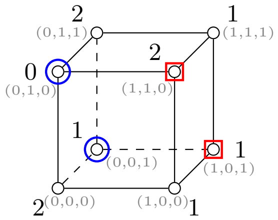
By summing the two inequalities and simplifying, we thus get that $f_N(\mathcal{H}_{[x=(p^{maj})_x \wedge y=(p^{maj})_y]}) \geq f_N(\mathcal{H}_{[y=1-(p^{maj})_y \wedge x=1-(p^{maj})_x]})$, implying that $f_N(P_1^{\mathbf{s}^{xy}}) \geq f_N(P_2^{\mathbf{s}^{xy}})$. Therefore, c_1 still wins in $\mathbf{s}^{xy}$, and thus this move was not a better response for c_2. Hence, $\mathbf{s}^0$ is a 2-local equilibrium. □

Notice that the condition of the theorem cannot be relaxed to require just that $\mathcal{H}_1 \cap \mathcal{H}^{maj} \neq \emptyset$ and $\mathcal{H}_2 \cap \mathcal{H}^{maj} \neq \emptyset$ separately. This is somewhat intuitive from the proof itself, and is formally stated in the following proposition.

Proposition 6 *A 2-local equilibrium may not exist in a BSC game even when $m = 2$, $K = 3$, and the sets of strategies are such that $\mathcal{H}_1 \cap \mathcal{H}^{maj} \neq \emptyset$ and $\mathcal{H}_2 \cap \mathcal{H}^{maj} \neq \emptyset$.*

Proof Consider a BSC game with $m = 2$ candidates, $n = 10$ voters and $K = 3$ issues. Let the sets of strategies be $\mathcal{H}_1 := \{(1, 1, 0), (1, 0, 1)\}$ and $\mathcal{H}_2 := \{(0, 1, 0), (0, 0, 1)\}$; these are shown (together with the distribution of voters) in the hypercube represented below (left) with red squares for $\mathcal{H}_1$ and blue circles for $\mathcal{H}_2$. The table below (right) reports all possible states of the game; the number of votes that each candidate gets is given for each state, and it is written in bold to represent the winner.

Notice that in this example $\mathcal{H}^{maj} = \mathcal{H}$ (the whole hypercube), thus both $\mathcal{H}_1 \cap \mathcal{H}^{maj} \neq \emptyset$ and $\mathcal{H}_2 \cap \mathcal{H}^{maj} \neq \emptyset$ hold. However, from each state there is an improving 2-local deviation, and so no 2-local equilibrium exists.



		$s_2 \in \mathcal{H}_2$		
		$(0,1,0)$		$(0,0,1)$
$s_1 \in \mathcal{H}_1$	$(1,1,0)$	$(\mathbf{5},5)$	$\xrightarrow{c_2}$	$(4,\mathbf{6})$
		$c_1 \uparrow$		$\downarrow c_1$
	$(1,0,1)$	$(4,\mathbf{6})$	$\xleftarrow{c_2}$	$(\mathbf{5},5)$

$\square$

Nevertheless, Theorem 5 allows for a simple corollary involving 1-local equilibria, as stated below.

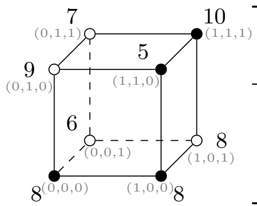
Corollary 7 *There always exists a 1-local equilibrium in a BSC game where $m = 2$, there is a position $p^* \in \mathcal{H}_1 \cap \mathcal{H}^{maj}$ and a strategy $s_2 \in \mathcal{H}_2$ such that $dist(p^*, s_2) = 1$.*

Proof This follows from Theorem 5, considering that any 1-local deviation of c_2 from the state $\mathbf{s}^0 = (p^*, s_2)$ corresponds to a 2-local deviation from $\tilde{\mathbf{s}}^0 = (p^*, p^*)$ (which is not necessarily admissible for the game).[6] As $\tilde{\mathbf{s}}^0$ would be a 2-local equilibrium, there cannot be any 1-local improving deviation for c_2 from $\mathbf{s}^0$. $\square$

However, Theorem 5 is tight in the sense that, under the same conditions, the positive result for the existence of a 2-local equilibrium cannot be extended to 3-local equilibria, as stated below.

Proposition 8 *A 3-local equilibrium may not exist in a BSC game even when $m = 2$, $K = 3$, and the sets of candidates' strategies coincide, contain $\mathcal{H}^{maj}$ and are connected.*

Proof Consider a BSC game with $m = 2$ candidates, $n = 61$ voters and $K = 3$ issues. The sets of strategies are $\mathcal{H}_1 = \mathcal{H}_2 = \{(0,0,0), (1,0,0), (1,1,0), (1,1,1)\}$. The distribution of voters on the hypercube is represented below (left), where the set of strategies of both candidates is marked by black vertices. In this game, $\mathcal{H}^{maj} = \{p^{maj}\}$ where $p^{maj} = (1,1,1)$. The table below (right) reports all possible states of the game; the number of votes that each candidate gets is given for each state, and it is written in bold to represent the winner. One can observe that, from each of these states, there is a 3-local deviation.



		$s_2 \in \mathcal{H}_2$			
		$(0,0,0)$	$(1,0,0)$	$(1,1,0)$	$(1,1,1)$
$s_1 \in \mathcal{H}_1$	$(0,0,0)$	$(\mathbf{30.5}, 30.5)$	$(30, \mathbf{31})$	$(30, \mathbf{31})$	$(\mathbf{31}, 30)$
	$(1,0,0)$	$(\mathbf{31}, 30)$	$(\mathbf{30.5}, 30.5)$	$(30, \mathbf{31})$	$(30, \mathbf{31})$
	$(1,1,0)$	$(\mathbf{31}, 30)$	$(\mathbf{31}, 30)$	$(\mathbf{30.5}, 30.5)$	$(30, \mathbf{31})$
	$(1,1,1)$	$(30, \mathbf{31})$	$(\mathbf{31}, 30)$	$(\mathbf{31}, 30)$	$(\mathbf{30.5}, 30.5)$

$\square$

[6] It may also correspond to not deviating from $\tilde{\mathbf{s}}^0$, but as c_1 wins in this state, it will never be an improving deviation for c_2.

		Positions				
		e_i for $i \in [3q]$	p^1	p^2	p^3	p^4
Strategies	$s^j \in \mathcal{H}_c$ for $c \in C_S$	$\begin{cases} 2 \text{ if } x_i \in S_j \\ 4 \text{ otherwise} \end{cases}$	5	5	4	4
	$s_a^1 \in \mathcal{H}_{c_a}$	3	2	2	3	1
	$s_a^2 \in \mathcal{H}_{c_a}$	3	0	4	3	3
	$s_b^1 \in \mathcal{H}_{c_b}$	3	4	0	1	1
	$s_b^2 \in \mathcal{H}_{c_b}$	3	2	2	1	3

Table 2: Distance between each possible strategy of candidates and positions of the hypercube containing voters.

Moreover, deciding about the existence of a Nash equilibrium, and even of a t-local equilibrium, for all $2 \leq t \leq K$, is computationally hard, as stated in the next theorem.

Theorem 9 *Deciding whether there exists a t-local equilibrium is NP-hard, for $t \in \{2, \dots, K\}$, even under narcissistic preferences.*

Proof We perform a reduction from Exact Cover by 3-Sets (X3C), a problem known to be NP-complete [30]. In an instance of X3C, we are given a set $X = \{x_1, x_2, \dots, x_{3q}\}$ and a set $S = \{S_1, S_2, \dots, S_r\}$ of 3-element subsets of X and we ask whether there exists an exact cover, i.e., a subset $S' \subseteq S$ such that every element of X occurs in exactly one member of S', in other words, S' is a partition of X. We construct a BSC game as follows. First, we consider $K = 3q + 4$ issues, and we create $(3q + 10)w_p + 23$ voters, given an arbitrary integer w_p such that $w_p > 24$, where the voters are distributed as follows on the positions of the hypercube:

- w_p voters on each position $e^i = (0, \dots, 0, 1, 0, \dots, 0)$ such that $e^i_i = 1$ and $e^i_j = 0$ for every $j \in [3q + 4] \setminus \{i\}$, for every $i \in [3q]$;
- $\frac{5}{2}w_p + 11$ voters on position $p^1 := (0, \dots, 0, 1, 1, 0, 0)$;
- 7 voters on position $p^2 := (0, \dots, 0, 0, 0, 1, 1)$;
- $\frac{5}{2}w_p + 3$ voters on position $p^3 := (0, \dots, 0, 0, 0, 1, 0)$;
- 2 voters on position $p^4 := (0, \dots, 0, 0, 0, 0, 1)$.

We create $q + 2$ candidates and denote the set of candidates by $C := C_S \cup \{c_a, c_b\}$, where the set $C_S := \bigcup_{j=1}^{q} c_j$ regroups the so-called subset-candidates. The sets of strategies are:

- $\mathcal{H}_c := \mathcal{H}_S := \bigcup_{j=1}^{r}\{s^j = (s_1, \dots, s_{3q}, 0, 0, 0, 0) \in \{0, 1\}^K : \forall i \in [3q],\ s_i = 1 \text{ iff } x_i \in S_j\}$ for every $c \in C_S$;
- $\mathcal{H}_{c_a} := \{s_a^1 := (0, \dots, 0, 1, 0, 0, 1), s_a^2 := (0, \dots, 0, 1, 1, 0, 0)\}$;
- $\mathcal{H}_{c_b} := \{s_b^1 := (0, \dots, 0, 0, 0, 1, 1), s_b^2 := (0, \dots, 0, 1, 0, 1, 0)\}$.

The candidates' truthful positions are arbitrary and their preferences are narcissistic. We report in Table 2 all distances between each possible candidate's strategy and the positions that correspond to the opinion of some voters; and in Table 3 the number of votes that candidates c_a and c_b can get from positions p^1, p^2, p^3, and p^4.

One can prove that there exists a Nash equilibrium in the BSC game iff there exists a subset of S that is a partition of X.

The idea is that only candidates c_a and c_b may have an incentive to deviate and they would do so only if there is a position e^i for $i \in [3q]$ not "covered" by the strategy position

		$\mathcal{H}_{c_b}$	
		s_b^1	s_b^2
$\mathcal{H}_{c_a}$	s_a^1	$(\frac{5}{2}w_p + 12, \frac{5}{2}w_p + 11)$	$(\frac{5}{4}w_p + 11, \frac{15}{4}w_p + 12)$
	s_a^2	$(\frac{5}{2}w_p + 11, \frac{5}{2}w_p + 12)$	$(\frac{5}{2}w_p + 12, \frac{5}{2}w_p + 11)$

Table 3: Number of votes, from the voters positioned at $\{p^1, p^2, p^3, p^4\}$, that candidates c_a and c_b get according to all their possible strategies.

of a subset-candidate. A better response for candidate c_a or c_b would trigger a cycle of local deviations, preventing a Nash equilibrium from existing, as it can be deduced from Table 3. Note that the only deviations that c_a or c_b can make are towards another strategy position at distance 2 from their previous strategy position. It follows from the connections between Nash and local equilibria that the complexity result also holds for t-local equilibria, for all $2 \leq t \leq K$.

Suppose first that there exists a subset $S' \subseteq S$ such that every element of X occurs in exactly one member of S', in other words, S' is a partition of X. Since $|X| = 3q$ and all elements of S are subsets of X of size 3, we have $|S'| = q$. Consider the profile of strategies s such that all the q subset-candidates choose strategies in $\mathcal{H}_S$ that are associated with the elements of S', i.e., $s_{C_S} = \{s^j : S_j \in S'\}$, and such that candidates c_a and c_b play strategies s_a^2 and s_b^1, respectively. As reported in Table 2 and because of the partition definition, each subset-candidate choosing a strategy s^j for $S_j \in S'$ is positioned at distance 2 of the three distinct positions e_i associated with variables $x_i \in S_j$, therefore they get all the associated voters (candidates c_a and c_b cannot be closer). It follows that each subset-candidate gets $3w_p$ voters. They cannot obtain more voters since, as reported in Table 2, no matter which strategy is chosen by candidates c_a and c_b, subset-candidates will never get closer to positions p^1, p^2, p^3, and p^4. Moreover, since no voter positioned in a position e_i is accessible for candidate c_a or c_b, we obtain that candidates c_a and c_b obtain exactly the voters on positions p^1, p^2, p^3, and p^4 that are closer to their respective strategies s_a^2 and s_b^1, i.e., as reported in Table 3, $\frac{5}{2}w_p + 11$ voters and $\frac{5}{2}w_p + 12$ voters, respectively. Since $w_p > 24$, we have $\frac{5}{2}w_p + 12 < 3w_p$, and thus the subset-candidate that is the most advantaged in the tie-breaking rule is winning the election with $3w_p$ votes.

Let us prove that strategy profile s is a Nash equilibrium. A subset-candidate cannot make the outcome better according to her preferences, because she cannot get more votes. Therefore, no subset-candidate has an incentive to change her strategy in s. The only new profiles that candidates c_a and c_b can reach from s, by a Nash deviation, are those where they play s_a^1 and s_b^1, respectively, or s_a^2 and s_b^2, respectively. However, they still cannot get voters from positions e_i for $i \in [3q]$ since, in the best case, they can make a distance of 3 and all positions e_i are covered by candidates at distance 2. It follows that the best new score that they can have is $\frac{5}{2}w_p + 12$ (see Table 3), which cannot change the winner. It follows that candidates c_a and c_b have no incentive to change their strategies neither, and thus s is a Nash equilibrium, and thus also a t-local equilibrium, for any $t \in [K]$.

Suppose now that there does not exist any subset of S that is a partition of X. It follows that the q candidates that compose C_S cannot choose strategies in $\mathcal{H}_S$ such that they can be at distance 2 of every position e_i for $i \in [3q]$. Therefore, there exists an index $i \in [3q]$ such that all candidates $c \in C_S$ are at distance 4 of e_i and thus candidates c_a and c_b are both at distance 3 of e_i, no matter which strategy they play (see Table 2). Consequently, candidates c_a and c_b share the voters of position e_i and receive both $\frac{w_p}{2}$ voters from this position. From Table 3 and the fact that no subset-candidate can get voters from positions p^1, p^2, p^3, and p^4, we have that no subset-candidate can get more than $3w_p$ voters whereas the best score for candidate c_a and c_b (depending on what they play) is at least $\frac{w_p}{2} + \frac{5}{2}w_p + 12 = 3w_p + 12$. It follows that the winner is necessarily candidate c_a or c_b in every possible strategy profile. Note that since candidates c_a and c_b are at the same distance 3 of every position e_i, they necessarily receive exactly the same number of voters from such positions, we can thus focus on what they receive from positions p^1, p^2, p^3, and p^4 (see Table 3). If candidate c_a plays strategy s_a^1 and candidate c_b strategy s_b^1, then candidate c_a is winning. It follows that candidate c_b has an incentive to move to strategy s_b^2 leading to a profile where she is winning. From that new profile, candidate c_a has an incentive to move to strategy s_a^2, leading to a profile where she is winning. From that new profile, candidate c_b has an incentive to come back to strategy s_b^1, leading to a profile where she is winning. Finally, from that new profile, candidate c_a has an incentive to come back to strategy s_a^1, leading to a profile where she is winning. It follows that, whatever the chosen strategies of c_a and c_b between their two possible ones, the non-winner between them always has an incentive to choose her other possible strategy to make herself the winner. Hence, there is no Nash equilibrium, nor a t-local equilibrium, for all $2 \leq t \leq K$. $\square$

The question is nevertheless open whether hardness still holds for 1-local equilibria or connected candidates' sets of strategies. Remark that there exists a fixed-parameter tractable algorithm w.r.t. the number of issues and candidates for deciding the existence of a t-local equilibrium, since it suffices to check all the possible states of the game (by the game's structure, checking whether a candidate has an improving Nash deviation may already take $\mathcal{O}(2^K)$ steps).

Nevertheless, positive results can be found when restrictions are added on the distribution of voters or on candidates' strategies.

4.2 Restrictions on the Distribution of Voters

Restricting to a single-peaked distribution of voters allows to guarantee the existence of a Nash equilibrium for two candidates, as long as any one of them can *use* the peak as their strategy.

Theorem 10 *There always exists a Nash equilibrium in a BSC game under a single-peaked distribution of voters when $m = 2$ and the peak position p^* is included in $\mathcal{H}_1 \cup \mathcal{H}_2$.*

This result aligns with the usual intuition: when a peak position exists, candidates can take advantage of it to win the election. To prove this intuitive fact, we will need several technical lemmas about single-peaked distributions. Namely, we first prove in the next lemma that, when a candidate positions herself on p^*, she will get the majority of votes in the election. The proof of this particular lemma is quite extensive, and it involves reducing the general problem to a case where both candidates are placed at antipodal positions (a setting where Hall's theorem can be applied).

Lemma 11 *Consider a BSC game with $m = 2$ candidates under a single-peaked distribution f_N with respect to $p^* \in \mathcal{H}$. Then, for any $p \in \mathcal{H}$, defining the state $\mathbf{s} = (s_i, s_{-i}) = (p^*, p)$, we have:*

$$f_N(P_i^{\mathbf{s}}) \geq f_N(P_{-i}^{\mathbf{s}})$$

i.e., she who takes the peak as her strategy will always have the most votes.

Proof First of all, if $p = p^*$, then we trivially have $f_N(P_i^{\mathbf{s}}) = f_N(P_{-i}^{\mathbf{s}}) = 0$, and our claim is satisfied. Let us thus assume from now on that $p \neq p^*$.

By Observation 3.1, for every position $p' \in \mathcal{H}$, the value p'_j on issue $j \in X_=^{\mathbf{s}}$ does not matter for distinguishing between the two sets of influence and the indifference set. Therefore, we can restrict our attention to issues in $X_{\neq}^{\mathbf{s}}$, where $r^{\mathbf{s}} = |X_{\neq}^{s}| = dist(s_1, s_2)$ (the superscript may be omitted). Consider any vector $a \in \{0,1\}^{X_=^{\mathbf{s}}}$, we may only consider the game given in the *restricted hypercube* $\mathcal{H}^a := \{p \in \mathcal{H} \ : \ \forall j \in X_=^{\mathbf{s}},\ p_j = a_j\}$. Positions p^* and p may not live in this set, but as we only care about the issues in $X_{\neq}^{\mathbf{s}}$, it will suffice to consider the positions $p^*|_a := \begin{cases} a_j & \forall j \in X_=^{\mathbf{s}} \\ p_j^* & \forall j \in X_{\neq}^{\mathbf{s}} \end{cases}$ and $p|_a$ defined similarly. Let us denote $\mathbf{s}^a := (p^*|_a, p|_a)$. We might notice that:

Claim 12 $f_N|_{\mathcal{H}^a} : \mathcal{H}^a \to \mathbb{N}$ *is single-peaked with respect to $p^*|_a$.*

Proof We will use the fact that $y \in [x, p^*]$ iff for every $i \in [K]$, $x_i = p_i^*$ implies $x_i = p_i^* = y_i$. Let $x \in \mathcal{H}^a$ be any position and $y \in [x, p^*|_a]$. Notice that, as for every $i \in X_=^{\mathbf{s}},\ x_i = p_i^* = a_i$, any position $y \in [x, p^*|_a]$ will satisfy that for every $i \in X_=^{\mathbf{s}},\ x_i = (p^*|_a)_i = y_i = a_i$ (in particular, $y \in \mathcal{H}^a$) and for every $i \in X_{\neq}^{\mathbf{s}},\ x_i = (p^*|_a)_i = p_i^*$ implies $x_i = p_i^* = (p^*|_a)_i = y_i$ by definition of the *betweeness* relation. This actually implies that $y \in [x, p^*]$, as for every $i \in [K]$, if $i \in X_{\neq}^{\mathbf{s}}$ we have $x_i = p_i^*$ which implies $x_i = p_i^* = y_i$; and for every $i \in X_=^{\mathbf{s}}$ either $p_i^* \neq a_i = x_i$ (and the condition is trivially satisfied) or $p_i^* = a_i$ and we have $x_i = p_i^* = y_i = a_i$. So, as f_N is single-peaked with respect to p^* and $y \in [x, p^*]$, $f_N(x) \leq f_N(y)$, i.e., we have proved that

$$\forall x \in \mathcal{H}^a,\ \forall y \in [x, p^*|_a],\ f_N(x) \leq f_N(y)$$

and so, $f_N|_{\mathcal{H}^a}$ is single-peaked with respect to $p^*|_a$. $\diamond$

One advantage of considering this restriction is that $\mathcal{H}^a \simeq \{0,1\}^r$; namely, it can be seen as a game involving only the r issues that *we care about* (i.e., those in $X_{\neq}^{\mathbf{s}}$). On the other hand, $p^*|_a$ and $p|_a$ are antipodal in this hypercube, as $dist(p^*|_a, p|_a) = dist_{\neq}(p^*, p) = r$. Furthermore, we can express the sets $P_i^{\mathbf{s}}$ in terms of their *restriction* to $\mathcal{H}^a$.

Claim 13 *For $i \in \{1, 2\}$, we have*

$$P_i^{\mathbf{s}} = \bigcup_{a \in \{0,1\}^{X_=^{\mathbf{s}}}} P_i^{\mathbf{s}^a}|_{\mathcal{H}^a}$$

Proof As $\mathcal{H} = \bigcup_{a\in\{0,1\}^{X^{\mathbf{s}}_{=}}} \mathcal{H}^a$, we have that $P_i^{\mathbf{s}} = \bigcup_{a\in\{0,1\}^{X^{\mathbf{s}}_{=}}} P_i^{\mathbf{s}} \cap \mathcal{H}^a$. However, by Observation 3.1, we also see that $P_i^{\mathbf{s}} \cap \mathcal{H}^a = \{x \in \mathcal{H}^a : \mathit{dist}_{\neq}(s_i, x) < \mathit{dist}_{\neq}(s_{-i}, x)\} = \{x \in \mathcal{H}^a : \mathit{dist}_{\neq}(s_i|_a, x) < \mathit{dist}_{\neq}(s_{-j}|_a, x)\}$, as $\mathit{dist}_{\neq}(\cdot,\cdot)$ is calculated only on the issues in $X^{\mathbf{s}}_{\neq}$. This last set corresponds precisely to the definition of $P_i^{\mathbf{s}^a}|_{\mathcal{H}^a}$, which allows us to conclude. $\diamond$

With this in mind, it will be enough to show that for every $a \in \{0,1\}^{X^{\mathbf{s}}_{=}}$, $f_N(P_i^{\mathbf{s}^a}|_{\mathcal{H}^a}) \geq f_N(P_{-i}^{\mathbf{s}^a}|_{\mathcal{H}^a})$. Fortunately, this is not too complicated, as $\mathcal{H}^a$ is a hypercube on r issues, with a distribution of voters $f_N|_{\mathcal{H}^a}$ that is single-peaked with respect to $p^*|_a$, and where $p^*|_a$ and $p|_a$ are antipodal.

We can partition each set of influence $P_i^{\mathbf{s}^a}$ for $i \in \{1,2\}$ in $d_{\mathbf{s}}+1$ layers (where $d_{\mathbf{s}}$ is the critical distance), defined as follows: $P_i^{\mathbf{s}^a}(\ell) := \{p' \in \mathcal{H}^a : \mathit{dist}(s_i, p') = \ell\}$ for each $\ell \in \{0,1,\ldots,d_{\mathbf{s}}\}$. By construction, $|P_i^{\mathbf{s}^a}(\ell)| = \binom{r}{\ell}$ for each $\ell \in \{0,1,\ldots,d_{\mathbf{s}}\}$ and $i \in \{1,2\}$ and, by Observation 3.2, $P_i^{\mathbf{s}^a}|_{\mathcal{H}^a} = \bigcup_{\ell=0}^{d_{\mathbf{s}}} P_i^{\mathbf{s}^a}(\ell)$. We construct, for each layer $\ell \in \{0,1,\ldots,d_{\mathbf{s}}\}$, a bipartite graph $G_\ell^a := (P_i^{\mathbf{s}^a}(\ell) \cup P_{-i}^{\mathbf{s}^a}(\ell), E^\ell)$ such that $\{p,p'\} \in E^\ell$ iff $p \in P_i^{\mathbf{s}^a}(\ell)$ is on a shortest path in $G^{\mathcal{H}^a}$ between $p' \in P_{-i}^{\mathbf{s}^a}(\ell)$ and $p^*|_a$.

We prove (see Claim 14) that G_ℓ^a is regular, implying that there exists a perfect matching $\varphi_\ell^a : P_{-i}^{\mathbf{s}^a}(\ell) \to P_i^{\mathbf{s}^a}(\ell)$ in G_ℓ^a, for every $\ell \in \{0,1,\ldots,d_{\mathbf{s}}\}$. Matching φ_ℓ^a assigns to each position $p' \in P_{-i}^{\mathbf{s}^a}(\ell)$ a position $\varphi_\ell^a(p') \in P_i^{\mathbf{s}^a}(\ell)$ such that $\varphi_\ell^a(p')$ is on a shortest path between p' and $p^*|_a$ in $\mathcal{H}^a$. Thus, by single-peakedness of $f_N|_{\mathcal{H}^a}$, $f_N(\varphi_\ell^a(p')) \geq f_N(p')$. In particular, as the different *layers* $P_{\cdot}^{\mathbf{s}^a}(\ell)$ of $P_{\cdot}^{\mathbf{s}^a}|_{\mathcal{H}^a}$ are disjoint, we can define a bijection $\varphi^a : P_{-i}^{\mathbf{s}^a}|_{\mathcal{H}^a} \to P_i^{\mathbf{s}^a}|_{\mathcal{H}^a}$ such that, for every $\ell \in \{0,\ldots,d_{\mathbf{s}}\}$, if $p' \in P_{-i}^{\mathbf{s}^a}(\ell)$, then $\varphi^a(p') := \varphi_\ell^a(p')$. This bijection still respects that, for every $p' \in P_{-i}^{\mathbf{s}^a}|_{\mathcal{H}^a}$, we have $f_N(\varphi^a(p')) \geq f_N(p')$ by definition. Furthermore, $\varphi^a(p|_a) = p^*|_a$, as the map acts *layer to layer*.

Naturally, we can now define the map $\varphi : P_{-i}^{\mathbf{s}} \to P_i^{\mathbf{s}}$ that takes $p' \in P_{-i}^{\mathbf{s}}$ to $\varphi(p') = \varphi^a(p')$ whenever $p'_{X^{\mathbf{s}}_{=}} = a$. Again, by definition, this will satisfy that, for all $p' \in P_{-i}^{\mathbf{s}}$, we have $f_N(\varphi(p')) \geq f_N(p')$; so, in particular, $f_N(P_i^{\mathbf{s}}) \geq f_N(P_{-i}^{\mathbf{s}})$, which is what we wanted.

Now, for the sake of completeness, let us verify our claim about G_ℓ^a.

Claim 14 *There always exists a perfect matching* $\varphi_\ell^a : P_{-i}^{\mathbf{s}^a}(\ell) \to P_i^{\mathbf{s}^a}(\ell)$ *in bipartite graph* G_ℓ^a *for every* $\ell \in \{0,1,\ldots,d_{\mathbf{s}}\}$ *and* $a \in \{0,1\}^{X^{\mathbf{s}}_{=}}$.

Proof Because $s_i|_a = p^*|_a$ and $s_{-i}|_a = p|_a$ are antipodal positions in $\mathcal{H}^a$, they differ on *every issue* (in $X^{\mathbf{s}}_{\neq}$). Now, denote by $X_{\neq}^{(p|_a,p')}$ the set of ℓ issues on which position $p' \in P_{-i}^{\mathbf{s}^a}(\ell)$ differs from $p|_a$. Clearly, as $p^*|_a$ and $p|_a$ are antipodal, for every $i \in X_{\neq}^{(p|_a,p')}$, we have $p'_i = (p^*|_a)_i$. Therefore, for a position $\tilde{p} \in P_i^{\mathbf{s}^a}(\ell)$ to be on a shortest path between $p' \in P_{-i}^{\mathbf{s}^a}(\ell)$ and $p^*|_a$, $\tilde{p}$ needs to have the same value as p' on the subset of issues $X_{\neq}^{(p|_a,p')}$. It follows that the ℓ issues on which such a position $\tilde{p}$ differs from $p^*|_a$ do not belong to $X_{\neq}^{(p|_a,p')}$ (for which $\tilde{p}$ and $p^*|_a$ have the same value). Thus, there are exactly $\binom{r-\ell}{\ell}$ positions in $P_i^{\mathbf{s}^a}(\ell)$ that are in a shortest path between p' and $p^*|_a$. Hence, every position vertex $p' \in P_{-i}^{\mathbf{s}^a}(\ell)$ has degree $\binom{r-\ell}{\ell}$ in the bipartite graph G_ℓ^a.

Conversely, for a given position $\tilde{p} \in P_i^{\mathbf{s}^a}(\ell)$, $\tilde{p}$ differs from $p^*|_a$ on exactly ℓ issues, denoted by $X_{\neq}^{(p^*|_a,\tilde{p})}$, on which its values are the same as those of $p|_a$. Therefore, to be on a shortest path between $p^*|_a$ and a position $p \in P^{\mathbf{s}^a}_{-i}(\ell)$, p' needs to have the same values as $\tilde{p}$ on $X_{\neq}^{(p^*|_a,\tilde{p})}$, which means that the ℓ issues on which it differs from $p|_a$ cannot be in $X_{\neq}^{(p^*|_a,\tilde{p})}$. It follows that there exist $\binom{r-\ell}{\ell}$ such positions p'. Hence, every position vertex $\tilde{p} \in P^{\mathbf{s}^a}_i(\ell)$ has degree $\binom{r-\ell}{\ell}$ in the bipartite graph G^a_ℓ.

Consequently, G^a_ℓ is a bipartite regular graph, which implies that there exists a perfect matching in G^a_ℓ (see, e.g., [31]). $\diamond$

Having tied the loose ends, we have completed the proof of Lemma 11. $\square$

Though Lemma 11 is enough for ensuring that a candidate favored by the tie-breaking rule will always win the election (making such state a Nash equilibrium), it does not allow to handle the case of the 'un-favored' candidate taking the peak. If votes were tied, maybe c_2 could lose the election, and it is not clear whether an equilibrium would exist in such situation. Fortunately, the following lemma shows that, under a single-peaked distribution, votes can only be tied with the peak position if the other candidate is also positioned at a peak position herself.

Lemma 15 *Let f_N be a single-peaked distribution with respect to $p^* \in \mathcal{H}$. Then, for any $p \in \mathcal{H}$,*

$$f_N\left(P_1^{(p^*,p)}\right) = f_N\left(P_2^{(p^*,p)}\right) \Longrightarrow p \text{ is a peak for } f_N$$

Proof First, let us consider the state $\mathbf{s} = (p^*, p)$. Let $\varphi : P_2^{\mathbf{s}} \to P_1^{\mathbf{s}}$ be any bijection such that for every $x \in P_2^{\mathbf{s}}$, we have $\varphi(x) \in [x, p^*]$ (by the proof of Lemma 11, we know there exists one).

By definition of single-peakedness, we must have that $f_N(x) \leq f_N(\varphi(x))$, for every $x \in P_2^{\mathbf{s}}$. Now, we actually have that for every $x \in P_2^{\mathbf{s}}$, $f_N(x) = f_N(\varphi(x))$. Indeed, if there was a position $\tilde{x} \in P_2^{\mathbf{s}}$ such that $f_N(\tilde{x}) < f_N(\varphi(\tilde{x}))$, then

$$f_N(P_2^{\mathbf{s}}) = f_N(\tilde{x}) + \sum_{\substack{x \in P_2^{\mathbf{s}} \\ x \neq \tilde{x}}} f_N(x) < f_N(\varphi(\tilde{x})) + \sum_{\substack{x \in P_2^{\mathbf{s}} \\ x \neq \tilde{x}}} f_N(\varphi(x)) = f_N(P_1^{\mathbf{s}})$$

which contradicts our hypothesis. So, we have that $f_N(x) = f_N(\varphi(x))$, for every $x \in P_2^{\mathbf{s}}$.

In particular, we can construct φ as in the previous lemma, such that for every $a \in \{0,1\}^{X^{\mathbf{s}}_=}$, we can define $\varphi^a : P_2^{\mathbf{s}^a}|_{\mathcal{H}^a} \to P_1^{\mathbf{s}^a}|_{\mathcal{H}^a}$ such that, for every $x \in P_2^{\mathbf{s}^a}|_{\mathcal{H}^a}$, we have $\varphi^a(x) \in [x, p^*|_a]$, and $\varphi^a(p|_a) = p^*|_a$.

We can actually prove that for every $x \in [p|_a, p^*|_a]$, we have $f_N(x) = f_N(p|_a)$. On the one hand, as $x \in [p|_a, p^*|_a]$, by single-peakedness (of $f_N|_{\mathcal{H}^a}$ with respect to $p^*|_a$), $f_N(p|_a) \leq f_N(x)$. For the other inequality, we notice that, as $\varphi(p|_a) = \varphi^a(p|_a) = p^*|_a$, we must have $f_N(p|_a) = f_N(p^*|_a)$ (by our previous observation). But as $p^*|_a \in [x, p^*|_a]$, by single-peakedness of $f_N|_{\mathcal{H}^a}$, we have $f_N(x) \leq f_N(p^*|_a) = f_N(p|_a)$. So, indeed, we showed that, for every $x \in [p|_a, p^*|_a]$, we have $f_N(x) = f_N(p|_a)$.

Actually, as $p|_a$ and $p^*|_a$ are *antipodal* in $\mathcal{H}^a$, this means that, for every $x \in \mathcal{H}^a$, $f_N(x) = f_N(p|_a)$ (as every position in $\mathcal{H}^a$ lies in $[p|_a, p^*|_a]$).

To conclude the proof, we will show that f_N is single-peaked with respect to p. That is, we now need to show that, for every $x \in \mathcal{H}$ and every $y \in [x, p]$, $f_N(x) \leq f_N(y)$. For that purpose, let $x \in \mathcal{H}$ and $y \in [x,p]$, and define $\tilde{x} := \begin{cases} x_j & \forall j \in X^{\mathbf{s}}_{=} \\ y_j & \forall j \in X^{\mathbf{s}}_{\neq} \end{cases}$. Let us denote by $\tilde{a} := (\tilde{x}_j)_{j \in X^{\mathbf{s}}_{=}}$. By definition, $x, \tilde{x} \in \mathcal{H}^{\tilde{a}}$, and so $f_N(x) = f_N(\tilde{x}) = f_N(p|_{\tilde{a}})$.

Furthermore, we notice that $\tilde{x}_j = y_j$ holds for every $j \in X^{\mathbf{s}}_{\neq}$, by definition. So, if $p^*_j = \tilde{x}_j$, then $p^*_j = \tilde{x}_j = y_j$ (trivially). Similarly, for every $j \in X^{\mathbf{s}}_{=}$, as $y \in [x,p]$ whenever $x_j = p_j$, then $x_j = p_j = y_j$. But as $j \in X^{\mathbf{s}}_{=}$, both $p_j = \tilde{p}_j$ and $\tilde{x}_j = x_j$ hold. Therefore, whenever $\tilde{x}_j = p^*_j$, we have $y_j = \tilde{x}_j = p^*_j$, i.e., we have proven that, for every $j \in [K]$, $\tilde{x}_j = p^*_j$ implies $\tilde{x}_j = p^*_j = y_j$, and by the characterization, this means $y \in [\tilde{x}, p^*]$. By single-peakedness of f_N with respect to p^*, we get that $f_N(y) \geq f_N(\tilde{x})$, and together with the fact that $f_N(x) = f_N(\tilde{x})$, we finally have proven that: $f_N(x) \leq f_N(y)$, for every $x \in \mathcal{H}$ and $y \in [x,p]$. □

Joining both Lemma 11 and Lemma 15, we have the following characterization:

Lemma 16 *Let f_N be a single-peaked distribution with respect to $p^* \in \mathcal{H}$. Let $p \in \mathcal{H}$ be any other position on the hypercube $\mathcal{H}$. Then, the following are equivalent:*

1. *p is a peak for f_N,*
2. *$f_N\left(P_1^{(p,s_2)}\right) \geq f_N\left(P_2^{(p,s_2)}\right)$, for every $s_2 \in \mathcal{H}$,*
3. *$f_N\left(P_1^{(p,p^*)}\right) = f_N\left(P_2^{(p,p^*)}\right)$.*

Proof By Lemma 11, we have $1 \implies 2$. By Lemma 15, we have $3 \implies 1$. For $2 \implies 3$, it suffices to see that, if we take $s_2 = p^*$, we get $f_N(P_1^{(p,p^*)}) \geq f_N(P_2^{(p,p^*)})$. For the other inequality, notice that p^* is a peak for the distribution, so it satisfies (by Lemma 11) $f_N(P_2^{(s_1,p^*)}) \geq f_N(P_1^{(s_1,p^*)})$, for every $s_1 \in \mathcal{H}$. In particular, taking $s_1 = p$ gives us the other inequality: $f_N(P_2^{(p,p^*)}) \geq f_N(P_1^{(p,p^*)})$. With this, we conclude: $f_N(P_1^{(p,p^*)}) = f_N(P_2^{(p,p^*)})$. □

Having laid down the technical lemmas in place, the conclusion of the theorem comes quite naturally.

Proof of Theorem 10 Given a BSC game with two candidates, where the distribution of voters is single-peaked w.r.t. a peak position $p^* \in \mathcal{H}$, such that $p^* \in \mathcal{H}_1 \cup \mathcal{H}_2$, we consider the following cases:

1. $p^* \in \mathcal{H}_1$. If c_1 is able to *choose the peak* as her strategy, then any state $\mathbf{s} = (s_1, s_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ where $s_1 = p^*$ is a Nash equilibrium for the game. This comes from Lemma 11, as we will have $f_N(P_1^{\mathbf{s}}) \geq f_N(P_2^{\mathbf{s}})$. This implies (as *indifferent voters* are perfectly split between both candidates), that $score(c_1) \geq score(c_2)$, and so by the tie-breaking rule, candidate c_1 is always the winner. In other words, c_2 cannot change the outcome by deviating to any other strategy (as in all states where $s_1 = p^*$, c_1 wins). Thus, for every state $\mathbf{s} = (p^*, s_2) \in \mathcal{H}_1 \times \mathcal{H}_2$, $\mathbf{s}$ is a Nash equilibrium.
2. If $p^* \in \mathcal{H}_2$, then by Lemma 11 we will have $f_N(P_2^{(s_1,p^*)}) \geq f_N(P_1^{(s_1,p^*)})$, for every $s_1 \in \mathcal{H}_1$. Now, we can further split into two cases:
   - $f_N(P_2^{(s_1,p)}) > f_N(P_1^{(s_1,p)})$, for every $s_1 \in \mathcal{H}_1$, and so, for any possible response $s_1 \in \mathcal{H}_1$ of c_1, we always get $score(c_2) > score(c_1)$, i.e., c_2 always wins the election. Therefore, c_1 cannot change the outcome by deviating to any another strategy, and any state $\mathbf{s} = (s_1, p^*) \in \mathcal{H}_1 \times \mathcal{H}_2$ is a Nash equilibrium for the game.

- There exists $\tilde{s}_1 \in \mathcal{H}_1$ such that $f_N(P_2^{(\tilde{s}_1,p^*)}) \leq f_N(P_1^{(\tilde{s}_1,p^*)})$. This means that $f_N(P_2^{(\tilde{s}_1,p^*)}) = f_N(P_1^{(\tilde{s}_1,p^*)})$, and so, by Lemma 15, actually $\tilde{s}_1$ is a peak for f_N. We are back to our first case, as c_1 can *choose the peak* $\tilde{s}_1$ as her strategy. By the same argument from above, for every $s_2 \in \mathcal{H}_2$, c_1 wins in state $(\tilde{s}_1, s_2) \in \mathcal{H}_1 \times \mathcal{H}_2$, and so every state of the form $(\tilde{s}_1, s_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ is a Nash equilibrium for the game.

Having considered all possibilities, we conclude that for every instance of a BSC game with a single-peaked distribution with respect to $p^* \in \mathcal{H}$, such that $p^* \in \mathcal{H}_1 \cup \mathcal{H}_2$, we can find a Nash equilibrium. □

Note that Theorem 10 also holds without the need of a tie-breaking rule if some strictness condition is assumed for the distribution.[7] A single-peaked distribution f_N is said to be *strict* if it has *exactly* one peak $p^* \in \mathcal{H}$ (i.e., it is unique).[8]

Corollary 17 *There always exists a Nash equilibrium in the BSC game, when $m = 2$, under a strictly single-peaked distribution of voters of (unique) peak p^*, such that $p^* \in \mathcal{H}_i \setminus \mathcal{H}_{-i}$ for some $i \in \{1, 2\}$.*

Proof The proof works all the same as before. Say $p^* \in \mathcal{H}_i$ for $i \in \{1, 2\}$, by Lemma 11, we have that, for any $\mathbf{s} = (s_i, s_{-i}) = (p^*, p) \in \mathcal{H}_i \times \mathcal{H}_{-i}$: $f_N(P_i^{\mathbf{s}}) \geq f_N(P_{-i}^{\mathbf{s}})$.

Now, as stated in the proof of the theorem (and in Lemma 15), if there was any $p \in \mathcal{H}_{-i}$ such that $f_N(P_i^{\mathbf{s}}) = f_N(P_{-i}^{\mathbf{s}})$, then p would be a peak for f_N. As the peak is unique (by our strictness hypothesis), this means $p = p^* \in \mathcal{H}_{-i}$, which is absurd (as we supposed $p^* \notin \mathcal{H}_{-i}$).

Therefore, it must be that for every $p \in \mathcal{H}_{-i}$, in state $\mathbf{s} = (s_i, s_{-i}) = (p^*, p) \in \mathcal{H}_i \times \mathcal{H}_{-i}$, we have: $f_N(P_i^{\mathbf{s}}) > f_N(P_{-i}^{\mathbf{s}})$; and so, c_i always wins the election, leaving c_{-i} with no possible improving deviation, making all such states Nash equilibria. □

The previous proofs highlight an interesting behavior of the single-peaked distribution: positioning alone on the peak position always guarantees to win the election.

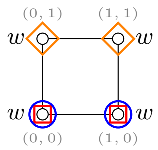
However, this positive result cannot be extended to more than two candidates since even a 1-local equilibrium may not exist.

Proposition 18 *A 1-local equilibrium may not exist in a BSC game even when $m = 3$, $K = 2$, the candidates' preferences are fixed, and the distribution of voters is uniform.*

Proof Consider a BSC game with $m = 3$ candidates, $K = 2$ issues and a number n of voters which is a multiple of 2^K. The voters are distributed in $\mathcal{H}$ in such a way that there are $w := \frac{n}{2^K}$ voters on each position $p \in \mathcal{H}$. The sets of strategies are $\mathcal{H}_1 = \mathcal{H}_2 = \{(0,0), (1,0)\}$ and $\mathcal{H}_3 = \{(0,1), (1,1)\}$. The distribution and the strategies are represented below on the left (red squares for $\mathcal{H}_1$, blue circles for $\mathcal{H}_2$, and orange diamonds for $\mathcal{H}_3$). The candidates' preferences are fixed and given below (right).

[7] Alternatively, the same result can be achieved by simply supposing that c_{-i} cannot choose *any* one of the peaks of the distribution as her strategy.

[8] This condition can easily follow from assuming, e.g., that either the peak is a *strict* maximum of the distribution ($\forall p \in \mathcal{H},\ f_N(p^*) > f_N(p)$) or that the *antipeak* is a strict minimum ($\forall p \in \mathcal{H}, f_N(\hat{p^*}) < f_N(p)$).



c_1:	c_1	$\succ$	c_3	$\succ$	c_2
c_2:	c_2	$\succ$	c_1	$\succ$	c_3
c_3:	c_3	$\succ$	c_2	$\succ$	c_1

The table below reports all possible states of the game; the number of votes that each candidate gets is given for each state, and it is written in bold to represent the winner. One can observe that, from each of these states, there is a 1-local deviation.

		$s_2 \in \mathcal{H}_2$					$s_2 \in \mathcal{H}_2$		
		$(0,0)$		$(1,0)$			$(0,0)$		$(1,0)$
$s_1 \in \mathcal{H}_1$	$(0,0)$	$(w, w, \mathbf{2w})$	$\xrightarrow{c_2}$	$(w, \frac{\mathbf{3}}{\mathbf{2}}\mathbf{w}, \frac{3}{2}w)$	$s_1 \in \mathcal{H}_1$	$(0,0)$	$(\frac{7}{6}w, \frac{7}{6}w, \frac{\mathbf{5}}{\mathbf{3}}\mathbf{w})$	$\xrightarrow{c_2}$	$(\frac{\mathbf{3}}{\mathbf{2}}\mathbf{w}, w, \frac{3}{2}w)$
		$c_1 \downarrow$		$\downarrow c_1$			$c_1 \uparrow$		$\uparrow c_1$
	$(1,0)$	$(\frac{\mathbf{3}}{\mathbf{2}}\mathbf{w}, w, \frac{3}{2}w)$	$\xleftarrow{c_2}$	$(\frac{7}{6}w, \frac{7}{6}w, \frac{\mathbf{5}}{\mathbf{3}}\mathbf{w})$		$(1,0)$	$(w, \frac{\mathbf{3}}{\mathbf{2}}\mathbf{w}, \frac{3}{2}w)$	$\xleftarrow{c_2}$	$(w, w, \mathbf{2w})$
		$s_3 = (0,1)$					$s_3 = (1,1)$		

□

4.3 Restrictions on Candidates' Strategies

The counterexample of Proposition 4 for the existence of a 1-local equilibrium is specific because the sets of candidates' strategies are disjoint and contain only two strategies. However, for two candidates and sets of strategies that coincide, there always exists a 2-local equilibrium, as stated more generally in the next theorem.

Theorem 19 *There always exists a 2-local equilibrium in a BSC game when $m = 2$ and $\mathcal{H}_2 \subseteq \mathcal{H}_1$. Such an equilibrium can be found in polynomial time.*

Proof If a majoritarian outcome $p^{maj} \in \mathcal{H}^{maj}$ belongs to $\mathcal{H}_1 \cap \mathcal{H}_2$ then, by Theorem 5, a 2-local equilibrium always exists. Therefore, we assume that $\mathcal{H}^{maj} \cap \mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$.

We will construct a particular sequence $s = \langle \mathbf{s}^0, \mathbf{s}^1, \ldots, \mathbf{s}^T \rangle$ of 2-local deviations and show that the constructed game dynamics must eventually converge to a 2-local equilibrium. We consider as the starting point of the sequence an arbitrary unanimous state s^0 such that $s^0 = (s_1^0, s_2^0)$ with $s_1^0 = s_2^0$ and $s_2^0 \in \mathcal{H}_2$. By the tie-breaking rule, candidate c_1 is the winner in s^0. If there does not exist a strategy $s_2^1 \in \mathcal{H}_2$ at distance at most two from s_2^0 such that candidate c_2 is the winner in the state $s^1 = (s_1^0, s_2^1)$, then state s^0 is a 2-local equilibrium, and we are done. Otherwise, we consider $s^1 = (s_1^0, s_2^1)$ as the next state in the sequence. From state s^1 where candidate c_2 is the winner, candidate c_1 has an incentive to join candidate c_2 on the same position strategy, leading to the next state $s^2 = (s_1^1, s_2^1)$ in the sequence where $s_1^1 = s_2^1$. Then, the same reasoning as in state s^0 applies on state s^2. Globally, we construct the sequence $s = (s^0, s^1, \ldots, s^T)$ such that each state s^t where t is even is unanimous with $s^t = (s_1^{t-1}, s_2^{t-1})$ such that $s_1^{t-1} = s_2^{t-1}$ and makes c_1 the winner, whereas each state s^t where t is odd is such that $s^t = (s_1^{t-1}, s_2^t)$ with $dist(s_1^{t-1}, s_2^t) \leq 2$ and makes c_2 the winner.

Let us consider a majoritarian position $p^{maj} \in \mathcal{H}^{maj}$. We can observe that no deviation in our constructed sequence can lead to go further from p^{maj}, i.e., deviations cannot choose issues' opinions which are opposite to those in p^{maj}. More precisely, (i) every 1-local deviation which changes the opinion on issue $x \in [K]$ must align with opinion $(p^{maj})_x$, and (ii) for every 2-local deviation where the candidate moves to opinions e_x and e_y on issues $x, y \in [K]$, respectively, where $e_x, e_y \in \{0,1\}$, we cannot have that both $e_x = 1 - (p^{maj})_x$ and $e_y = 1 - (p^{maj})_y$ hold. Let us prove these claims. Note that it suffices to focus on candidates c_2's deviations since afterward candidate c_1 reproduces the same deviation.

(i) Suppose that candidate c_2 moves to opinion $e_x = 1 - (p^{maj})_x$ on issue $x \in [K]$ to reach state s^t with t odd. Since the deviation is from a unanimous state, the positions of c_1 and c_2 in s^t only differ on issue x. For the deviation of c_2 to be an improving move, we need that $f_N(\mathcal{H}_{[x=e_x]}) > f_N(\mathcal{H}_{[x=1-e_x]})$, a contradiction with the majoritarian view of p^{maj}, i.e., $f_N(\mathcal{H}_{[x=(p^{maj})_x]}) \geq f_N(\mathcal{H}_{[x=1-(p^{maj})_x]})$.

(ii) Suppose that candidate c_2 moves to opinion e_x and e_y on issues $x, y \in [K]$, respectively, where $e_x = 1 - (p^{maj})_x$ and $e_y = 1 - (p^{maj})_y$, to reach state s^t with t odd. Since the deviation is from a unanimous state, the positions of c_1 and c_2 in s^t only differ on issues x and y. Therefore, for the deviation of c_2 to be an improving move, we need that $f_N(\mathcal{H}_{[x=e_x \wedge y=e_y]}) > f_N(\mathcal{H}_{[x=1-e_x \wedge y=1-e_y]})$. However, by decomposition of sets and definition of p^{maj}, we have:

$$\begin{aligned} & f_N(\mathcal{H}_{[x=(p^{maj})_x \wedge y=(p^{maj})_y]}) + f_N(\mathcal{H}_{[x=(p^{maj})_x \wedge y=1-(p^{maj})_y]}) \\ & \qquad \geq f_N(\mathcal{H}_{[x=1-(p^{maj})_x \wedge y=(p^{maj})_y]}) + f_N(\mathcal{H}_{[x=1-(p^{maj})_x \wedge y=1-(p^{maj})_y]}) \\ \text{and} \quad & f_N(\mathcal{H}_{[y=(p^{maj})_y \wedge x=(p^{maj})_x]}) + f_N(\mathcal{H}_{[y=(p^{maj})_y \wedge x=1-(p^{maj})_x]}) \\ & \qquad \geq f_N(\mathcal{H}_{[y=1-(p^{maj})_y \wedge x=(p^{maj})_x]}) + f_N(\mathcal{H}_{[y=1-(p^{maj})_y \wedge x=1-(p^{maj})_x]}). \end{aligned}$$

By summing the two inequalities and simplifying them, we thus get that $f_N(\mathcal{H}_{[y=(p^{maj})_y \wedge x=(p^{maj})_x]}) \geq f_N(\mathcal{H}_{[x=1-(p^{maj})_x \wedge y=1-(p^{maj})_y]})$, contradicting the improving move of c_2.

It follows that only three types of 2-local deviations are allowed:

1. 1-local deviations to an opinion $e_x = (p^{maj})_x$ on issue $x \in [K]$,
2. 2-local deviations to opinions $e_x = (p^{maj})_x$ and $e_y = (p^{maj})_y$ on issues $x, y \in [K]$,
3. 2-local deviations to opinions $e_x = (p^{maj})_x$ and $e_y = 1 - (p^{maj})_y$ on issues $x, y \in [K]$.

Suppose, for the sake of contradiction, that the constructed dynamics contains a cycle, i.e., the sequence $s = (s^0, s^1, \dots)$ is infinite and there exist two states s^t and $s^{t'}$ where $t < t'$ such that $s^t = s^{t'}$. Since candidate c_1 only reproduces the deviations of c_2, let us focus on the different strategies $\mu := (\mu_2^0, \mu_2^1, \dots, \mu_2^k) \subseteq \mathcal{H}^k$ of candidate c_2 in the cycle, where $\mu_2^0 = \mu_2^k$. Since issues are binary, the opinions on changed issues $x \in [K]$ are alternating between $(p^{maj})_x$ and $1 - (p^{maj})_x$ in sequence μ, and we need the same number of deviations to each issue opinion, for changed issues during the cycle. Say that, in the cycle, there are a moves of type 1, b moves of type 2, and c moves of type 3, with a, b, c non-negative integers. It follows from their definition that, among the changed issues of μ, there are $a + 2b + c$ issue opinions similar to p^{maj}, and c issue opinions opposite to p^{maj}. Since they must be equal by definition of the cycle, we have that $a + 2b + c = c$ and thus $a = b = 0$, implying that no move of type 1 or 2 can occur in the cycle.

To summarize, in the cycle μ of deviating positions taken by candidate c_2, candidate c_2 has only performed deviations of type 3. In such deviations, c_2 only changes the value of two issues x and y, one in the direction of p^{maj} and the other in the opposite direction, i.e., the new issue opinions are $e_x = (p^{maj})_x$ and $e_y = 1-(p^{maj})_y$, respectively. Let us construct a directed graph $G = (V, E)$, where $V \subseteq [K]$ is the set of the changed issues during μ, and there exists an arc $(i, j) \in E$ iff there exists a deviation between states of μ where candidate c_2 changes the values of issues i and j by choosing the new issue opinions $e_i = (p^{maj})_i$ and $e_j = 1-(p^{maj})_j$, respectively. By the fact that we have the same number of deviations to each issue opinion, for changed issues during the cycle, the in-degree of each vertex in G is equal to its out-degree. It follows that there exists a directed cycle in G, say $(x_1, x_2, \ldots, x_k) \subseteq [K]$. The deviations associated with this cycle are all deviations to change the opinions on issues x_ℓ and $x_{\ell+1}$, with $k+1=1$, for all $\ell \in [k]$, and the associated inequalities for these deviations to be improving moves are $f_N(\mathcal{H}_{[x_\ell=(p^{maj})_{x_\ell} \wedge x_{\ell+1}=1-(p^{maj})_{x_{\ell+1}}]}) > f_N(\mathcal{H}_{[x_\ell=1-(p^{maj})_{x_\ell} \wedge x_{\ell+1}=(p^{maj})_{x_{\ell+1}}]})$.

Let us prove that every position p which is not uniform on issues $x_1, \ldots, x_k$ of the directed cycle, i.e., there exist $i, j \in [k]$ where $(p)_{x_i} = (p^{maj})_{x_i}$ and $(p)_{x_j} = 1 - (p^{maj})_{x_j}$, appears exactly the same number of times on the left-hand side and on the right-hand side of inequalities associated with the cycle. Note that a position p' that is uniform on issues $x_1, \ldots, x_k$ cannot appear in any side of the associated inequalities by definition of the deviations and the cycle. Say that position p contains $t \geq 1$ maximal blocks of consecutive issues x_ℓ such that $(p)_{x_\ell} = (p^{maj})_{x_\ell}$ (x_k and x_1 are considered consecutive). By the cycle, position p must contain the same number t of maximal blocks of consecutive issues x_ℓ such that $(p)_{x_\ell} = 1 - (p^{maj})_{x_\ell}$. For every maximal block of consecutive issues $x_{\ell_1}, \ldots, x_{\ell_p}$ in p such that $(p)_{x_{\ell_i}} = (p^{maj})_{x_{\ell_i}}$ for $i \in [p]$, we have that $p \in \mathcal{H}_{[x_{\ell_p}=(p^{maj})_{x_{\ell_p}} \wedge x_{\ell_p+1}=1-(p^{maj})_{x_{\ell_p+1}}]}$, therefore p appears on the left-hand side of the inequality associated with the deviation on issues x_{ℓ_p} and x_{ℓ_p+1}. Moreover, because of the same block of consecutive issues, we have that $p \in \mathcal{H}_{[x_{\ell_1}=(p^{maj})_{x_{\ell_1}} \wedge x_{\ell_1-1}=1-(p^{maj})_{x_{\ell_1-1}}]}$, therefore p appears on the right-hand side of the inequality associated with the deviation on issues x_{ℓ_1} and x_{ℓ_1-1}. Since it holds for each maximal block and p cannot appear in inequalities associated with deviations on issues that are part of the same block for p, it follows that p appears the same number of times on left-hand sides and right-hand sides of inequalities associated with the cycle. Therefore, by summing all inequalities associated with the cycle and simplifying them, all members of the inequalities cancel and we get that $0 > 0$, a contradiction. □

This positive result is tight in the sense that the same conditions are not sufficient to guarantee the existence of 3-local equilibria, as it can be observed in Proposition 8. Beyond the connections between sets of candidates' strategies, another type of restriction that can be considered concerns the structure of these sets. In particular, we will provide positive results for the existence of 1-local equilibria with two candidates for the case where strategy sets are balls of radius one. Under this restriction, in the next two lemmas, we first determine conditions on the distance between the truthful strategies of the two candidates which allow for the existence of a 1-local equilibrium.

Lemma 20 *Suppose we are given a BSC game where $m = 2$ and candidates' strategies are balls of radius one. Consider a state $\mathbf{s}^0 = (p^{c_1}, p^{c_2})$ where candidate c_i wins for some $i \in \{1, 2\}$. If $r = dist(s_1^0, s_2^0)$ is even and there exists an issue $x \in X_{=}^{\mathbf{s}^0}$ on which candidate c_{-i} can change her opinion from s_{-i}^0 to perform an improving 1-local deviation from $\mathbf{s}^0$, then we can construct a 1-local equilibrium.*

Proof Consider an issue $x \in X_{=}^{\mathbf{s}^0}$. It means that both candidates have the same value $e_x \in \{0, 1\}$ on issue x in state $\mathbf{s}^0$, and that candidate c_{-i} goes further from c_i by deviating from $\mathbf{s}^0$ to a position strategy s_{-i}^{1x} such that $(s_{-i}^{1x})_j = (s_{-i}^0)_j$ for all $j \in [K] \setminus \{x\}$ and $(s_{-i}^{1x})_x = 1 - e_x$, leading to state $\mathbf{s}^{1x} = (s_{-i}^{1x}, s_i^0)$. After this 1-local deviation, consider the state $\mathbf{s}^{2x}$ which results from the 1-local deviation of candidate c_i on the same issue x, i.e., $\mathbf{s}^{2x} = (s_i^{2x}, s_{-i}^{1x})$ where $(s_i^{2x})_j = (s_i^0)_j$ for every $j \in [K] \setminus \{x\}$ and $(s_i^{2x})_x = 1 - e_x$.

By Lemma 1, $P_i^{\mathbf{s}^{2x}} = P_i^{\mathbf{s}^{1x}} \setminus (L_i^{\mathbf{s}^{1x}}{}_{[x=e_x]})$ and $P_{-i}^{\mathbf{s}^{2x}} = P_{-i}^{\mathbf{s}^{1x}} \setminus (L_{-i}^{\mathbf{s}^{1x}}{}_{[x=1-e_x]})$. Note that, by the previous move, we have $I_{[x=(s_i^0)_x]}^{\mathbf{s}^0} \subseteq (L_i^{\mathbf{s}^{1x}})_{[x=e_x]}$ and $I_{[x=1-(s_i^0)_x]}^{\mathbf{s}^0} \subseteq (L_{-i}^{\mathbf{s}^{1x}})_{[x=1-e_x]}$. If there exists a position $p \in (L_i^{\mathbf{s}^{1x}})_{[x=e_x]} \setminus I_{[x=(s_i^0)_x]}^{\mathbf{s}^0}$ then, by Lemma 1, p belongs to $P_i^{\mathbf{s}^0}$. Since $p \in L_i^{\mathbf{s}^{1x}}$, we know that $dist(p, s_{-i}^{1x}) - dist(p, s_i^0) = 1$. Since only a 1-local deviation is done from $\mathbf{s}^0$ to $\mathbf{s}^{1x}$, we must have $dist(p, s_{-i}^0) - dist(p, s_i^0) = 2$, and this difference, in distance between p and each of the candidates' position strategies, has decreased after the deviation of candidate c_{-i}, i.e., $p_x = 1 - e_x$, which is a contradiction. Hence, $I_{[x=(s_i^0)_x]}^{\mathbf{s}^0} = (L_i^{\mathbf{s}^{1x}})_{[x=e_x]}$. A similar reasoning can be applied to prove that $I_{[x=1-(s_i^0)_x]}^{\mathbf{s}^0} = (L_{-i}^{\mathbf{s}^{1x}})_{[x=1-e_x]}$.

Consequently, $P_i^{\mathbf{s}^{2x}} = P_i^{\mathbf{s}^0}$ and $P_{-i}^{\mathbf{s}^{2x}} = P_{-i}^{\mathbf{s}^0}$, implying that candidate c_i is winning in state s^{2x}. Now, by the shape of candidates' strategies, the only possible 1-local deviation from $\mathbf{s}^{2x}$ for candidate c_{-i} is to come back to position strategy s_{-i}^0 by reversing her value on issue x, leading to a new state $\mathbf{s}^{3x} = (s_i^{2x}, s_{-i}^0)$. By Lemma 1, $P_i^{\mathbf{s}^{3x}} = P_i^{\mathbf{s}^0} \cup I_{[x=1-e_x]}^{\mathbf{s}^0}$ and $P_{-i}^{\mathbf{s}^{3x}} = P_{-i}^{\mathbf{s}^0} \cup I_{[x=e_x]}^{\mathbf{s}^0}$. We know that $f_N(P_i^{\mathbf{s}^0}) \geq_{\rhd} f_N(P_{-i}^{\mathbf{s}^0})$ (candidate c_i wins in $\mathbf{s}^0$) and that $f_N(I_{[x=1-e_x]}^{\mathbf{s}^0}) > f_N(I_{[x=e_x]}^{\mathbf{s}^0})$ (candidate c_{-i} wins in $\mathbf{s}^{1x}$). This implies that $f_N(P_i^{\mathbf{s}^{3x}}) > f_N(P_{-i}^{\mathbf{s}^{3x}})$ and thus candidate c_i is still winning in state $\mathbf{s}^{3x}$. Therefore, this last move is not improving for candidate c_{-i} and hence state s^{2x} is a 1-local equilibrium. □

Lemma 21 *Suppose we are given a BSC game where $m = 2$ and candidates' strategies are balls of radius one. Consider a state $\mathbf{s}^0 = (p^{c_1}, p^{c_2})$ where candidate c_i wins for some $i \in \{1, 2\}$. If $r = dist(s_1^0, s_2^0)$ is odd and there exists an issue $x \in X_{\neq}^{\mathbf{s}^0}$ on which candidate c_{-i} can change her opinion from s_{-i}^0 to perform an improving 1-local deviation from $\mathbf{s}^0$, then we can construct a 1-local equilibrium.*

Proof It means that the two candidates have different values on issue x in state $\mathbf{s}^0$, i.e., $(s_{-i}^0)_x = e_x$ and $(s_i^0)_x = 1 - e_x$ for $e_x \in \{0, 1\}$, and that candidate c_i goes closer to c_i by deviating from $\mathbf{s}^0$ to a position strategy s_{-i}^{1x} such that $(s_{-i}^{1x})_x = 1 - e_x$, leading to state $\mathbf{s}^{1x} = (s_{-i}^{1x}, s_i^0)$. Consider now state $\mathbf{s}^{2x}$ which is the same as $\mathbf{s}^{1x}$ except that candidate c_i reverses her value on issue x, i.e., $\mathbf{s}^{2x} = (s_i^{2x}, s_{-i}^{1x})$ where $(s_i^{2x})_j = (s_i^0)_j$ for every $j \in [K] \setminus \{x\}$ and $(s_i^{2x})_x = e_x$.

By Lemma 1, $P_i^{\mathbf{s}^{2x}} = P_i^{\mathbf{s}^{1x}} \cup I_{x=e_x}^{\mathbf{s}^{1x}}$ and $P_{-i}^{\mathbf{s}^{2x}} = P_{-i}^{\mathbf{s}^{1x}} \cup I_{x=1-e_x}^{\mathbf{s}^{1x}}$. However, by construction, $I_{x=e_x}^{\mathbf{s}^{1x}} = (L_{-i}^{\mathbf{s}^0})_{[x=(s_{-i}^0)_x]}$ and $I_{x=1-e_x}^{\mathbf{s}^{1x}} = (L_i^{\mathbf{s}^0})_{[x=1-(s_{-i}^0)_x]}$, and we know that $f_N((L_{-i}^{\mathbf{s}^0})_{[x=(s_{-i}^0)_x]}) < f_N((L_i^{\mathbf{s}^0})_{[x=1-(s_{-i}^0)_x]})$ (candidate c_{-i} is winning in state $\mathbf{s}^{1x}$). Therefore, $f_N(I_{x=e_x}^{\mathbf{s}^{1x}}) < f_N(I_{x=1-e_x}^{\mathbf{s}^{1x}})$. In addition with the fact that $f_N(P_i^{\mathbf{s}^{1x}}) < f_N(P_{-i}^{\mathbf{s}^{1x}})$ (candidate c_{-i} is winning in state $\mathbf{s}^{1x}$), we get that $f_N(P_i^{\mathbf{s}^{2x}}) < f_N(P_{-i}^{\mathbf{s}^{2x}})$, implying that candidate c_{-i} is still winning in state $\mathbf{s}^{2x}$.

It follows that candidate c_{-i} has no incentive to deviate from $\mathbf{s}^{2x}$ while the only possible 1-local deviation from $\mathbf{s}^{2x}$ for candidate c_i would lead to state $\mathbf{s}^{1x}$, where c_{-i} is the winner, therefore this move is not improving for c_i. Hence $\mathbf{s}^2$ is a 1-local equilibrium. □

The conditions stated in the two previous lemmas help us to establish the general existence of a 1-local equilibrium when strategy sets are balls of radius one, as derived in the next theorem.

Theorem 22 *There always exists a 1-local equilibrium in a BSC game when $m = 2$ and candidates' strategies are balls of radius one. Such an equilibrium can be found in polynomial time.*

Proof Consider the truthful state $\mathbf{s}^0 = (s_1^0, s_2^0)$ where $s_1^0 = p^{c_1}$ and $s_2^0 = p^{c_2}$. Say that c_i wins in $\mathbf{s}^0$ for some $i \in \{1, 2\}$.

Suppose first that there exists a strategy $s_{-i}^1 \in \mathcal{H}_{-i}$ such that c_i wins in state $\mathbf{s}^1 := (s_i^0, s_{-i}^1)$. The only 1-local deviation that c_{-i} could perform from $\mathbf{s}^1$ is towards her truthful strategy s_{-i}^0. However, this deviation is not a better response because it leads to $\mathbf{s}^0$, thus $\mathbf{s}^1$ is a 1-local equilibrium.

Hence, from now on, suppose that all possible 1-local deviations of c_{-i} from $\mathbf{s}^0$ lead to a state where c_{-i} wins. Denote by $\mathbf{s}^{1x}$ the state resulting from the deviation from $\mathbf{s}^0$ where c_{-i} changes her strategy s_{-i}^0 only on issue x. Consider the state $\mathbf{s}^{2x}$ which is the same as $\mathbf{s}^{1x}$ except that c_i changes her strategy s_i^0 only on issue x.

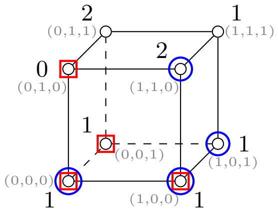
Suppose that $r := dist(s_i^0, s_{-i}^0)$ is even. If there exists an issue $x \in X_{=}^{\mathbf{s}^0}$ then, by Lemma 20, $\mathbf{s}^{2x}$ is a 1-local equilibrium. Let us thus assume, for r even, that all issues are in $X_{\neq}^{\mathbf{s}^0}$. This implies that s_i^0 and s_{-i}^0 are antipodal positions. By Lemma 1, among the $n^0 := f_N(I^{\mathbf{s}^0})$ voters positioned in $I^{\mathbf{s}^0}$, we must have strictly more than $\frac{n^0}{2}$ voters whose opinion has a value equal to $(s_i^0)_x$ on issue x, for all $x \in [K]$. In the same time, by Observation 3.2, the position of each such voter must have exactly $\frac{K}{2}$ issues with the same value as s_i^0, because they belong to $I^{\mathbf{s}^0}$. By the pigeonhole principle, these two requirements cannot be simultaneously fulfilled, a contradiction.

Suppose now that r is odd. If there exists an issue $x \in X_{\neq}^{\mathbf{s}^0}$ then, by Lemma 21, $\mathbf{s}^{2x}$ is a 1-local equilibrium. Let us thus assume, for r odd, that all issues are in $X_{=}^{\mathbf{s}^0}$. It follows that the sets of strategies of both candidates coincide, thus we can use the proof of Theorem 19 to construct a 1-local equilibrium, concluding the proof. □

The previous positive result for the existence of t-local equilibria when $t = 1$ cannot be extended to larger t, as stated below.

Proposition 23 *A 2-local equilibrium may not exist in a BSC game, even when $m = 2$, $K = 3$, and both candidates' strategies are balls of radius one.*

Proof Consider a BSC game with $m = 2$ candidates, $n = 9$ voters, and $K = 3$ issues. The sets of strategies are $\mathcal{H}_1 := \{(0,0,0), (0,1,0), (0,0,1), (1,0,0)\}$ and $\mathcal{H}_2 := \{(1,0,0), (0,0,0), (1,1,0), (1,0,1)\}$, which are balls of radius one around truthful positions $p^{c_1} = (0,0,0)$ and $p^{c_2} = (1,0,0)$, respectively. The distribution of voters and the sets of candidates' strategies (red squares for $\mathcal{H}_1$ and blue circles for $\mathcal{H}_2$) are represented below (left). The table below (right) reports all possible states of the game; the number of votes that each candidate gets is given for each state, and it is written in bold to represent the winner. From each of these states, there is a 2-local deviation.



		$s_2 \in \mathcal{H}_2$			
		$(1,0,0)$	$(0,0,0)$	$(1,0,1)$	$(1,1,0)$
$s_1 \in \mathcal{H}_1$	$(0,0,0)$	$(4, \mathbf{5})$	$(\mathbf{4.5}, 4.5)$	$(4, \mathbf{5})$	$(4, \mathbf{5})$
	$(1,0,0)$	$(\mathbf{4.5}, 4.5)$	$(\mathbf{5}, 4)$	$(4, \mathbf{5})$	$(4, \mathbf{5})$
	$(0,1,0)$	$(\mathbf{4.5}, 4.5)$	$(\mathbf{5}, 4)$	$(\mathbf{5}, 4)$	$(4, \mathbf{5})$
	$(0,0,1)$	$(\mathbf{4.5}, 4.5)$	$(\mathbf{5}, 4)$	$(4, \mathbf{5})$	$(\mathbf{5}, 4)$

□

5 Empirical Study of Local Equilibria

We also perform an experimental study on synthetic data in order to investigate the behavior of local equilibria in practice. In particular, we will perform two types of analysis: on the equilibria themselves and on the dynamics of local deviations. In general, we randomly generate 1,000 instances of BSC games with 5,000 voters whose positions are selected via a uniform distribution over the hypercube of issues (the number of voters does not impact the experiments, if it is large enough, since it only affects the scale of the "weights" associated with each position). The candidates' strategies are randomly generated in one of the following ways:

- *Random subsets*: For each candidate $c \in C$, first a *set size* $s \in [2^K]$ is uniformly sampled and then $\mathcal{H}_c$ is constructed by uniformly sampling exactly s random positions in $\mathcal{H}$ (also randomly defining p^c).
- *Connected subsets*: For each candidate $c \in C$, p^c is sampled uniformly at random from $\mathcal{H}$. Then, a *maximal radius* $b \in [K]$ is randomly (uniformly) sampled, and from it a desired set size s is selected uniformly from $[\sum_{k=1}^{b} \binom{K}{k}]$ (i.e., at most the size of a ball of radius b centered at p^c). Having set these values, either a depth-first-search (*DFS*) or a breadth-first-search (*BFS*) algorithm is used to construct the set $\mathcal{H}_c$ incrementally: neighbor by neighbor, ensuring connectedness. The end result of this procedure is a set $\mathcal{H}$ containing *exactly s positions* within a radius b of p^c.[9]
- *Random balls*: For each candidate $c \in C$, the set $\mathcal{H}_c$ is generated by uniformly sampling both a random position p^c in $\mathcal{H}$ and a *radius* b in $[K]$. Then, $\mathcal{H}_c$ is set as the ball of radius b around p^c.

All experiments are run considering both types of *candidate's preferences* mentioned in Section 3.2: *fixed preferences* (uniformly selected at random) and *narcissistic preferences* (which are deterministic), in order to compare their qualitative behavior.

5.1 Existence of Local Equilibria

We first analyze how frequently local equilibria exist and the proportion of states that are local equilibria. We generate BSC games for a number of issues $K \in \{3, 4, 5\}$ and

[9]Though this method for constructing connected sets might seem odd, it was preferred over the more intuitive construction via a *sequence of step-by-step random choices*, as the latter one has an inherent *bias* towards making large/small sizes of $\mathcal{H}_c$ very unlikely.

a number of candidates $m \in \{2, 3, 4\}$. For each set of parameters, Figure 1 presents the proportion of games, over the 1,000 ones that were generated, that admit a t-local equilibrium, for each $t \in [K]$.

The most noteworthy observation from Figure 1, which contrasts with our theoretical results exhibiting several negative results, is that a t-local equilibrium almost always exists for every $t \in [K]$. Indeed, for all sets of experiments under consideration, the frequency of existence is around 95% and is also very often close to 100%. In accordance with the theoretical connection between t-local equilibria, stating that a t-local equilibrium is also a t'-local equilibrium for $t' \leq t$, we observe that the frequency of existence of 1-local equilibria is greater than the frequency of existence of 2-local equilibria, and so on.

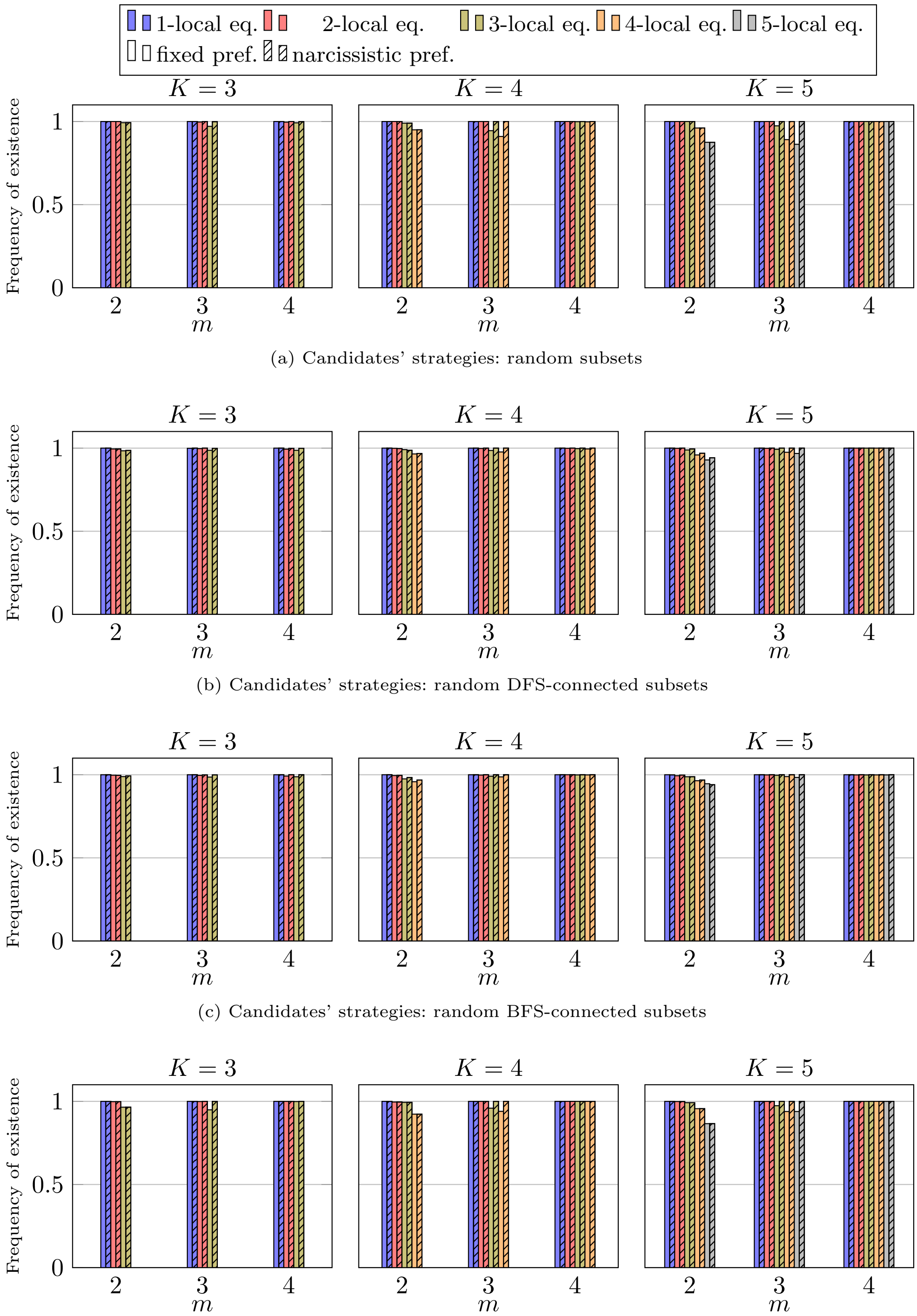
An interesting observation is that for the *random balls* strategy sets, a 1-local equilibrium could always be found in the generated games, raising the question of whether a deeper theoretical result (similar to Theorem 22) might be at play. However, this is not the case for every kind of strategy set (notably the *connected-DFS variant*), where examples without any kind of t-local equilibria, though extremely uncommon, could also be found via simulations.

We know that an equilibrium under fixed candidates' preferences is also stable under narcissistic preferences. This fact is clearly visible in Figure 1 since the frequency of existence is always greater for narcissistic preferences. Interestingly, in our experiments, all games with at least 3 candidates admit a 1-local equilibrium under narcissistic preferences (raising again the question about related theoretical guarantees).

Now, we investigate how many states are equilibria. More precisely, by generating all possible states, we verify whether each one is a t-local equilibrium for each $t \in [K]$ and then we compute the average proportion of states that are t-local equilibria over all of the 1,000 generated games. The results are presented in Figure 2.

As for the question of existence, we can again see that, as expected, there are more 1-local equilibria than 2-local equilibria, and so on. For every set of parameters, the proportion of t-local equilibria is rather close for all $t \in \{2, \ldots, K\}$. However, a remarkable difference can be seen between the proportion of 1-local equilibria and the proportion of 2-local equilibria: 1-local equilibria are around 1.5 times more common (even more than twice as common in the *random balls* setting). This particular behavior of 1-local equilibria is already notable in our theoretical results since our counterexamples for the existence of a Nash equilibrium are typically already counterexamples for the existence of 2-local equilibria.

Note that the number of t-local equilibria under narcissistic preferences is around twice that number under fixed preferences, for all sets of experiments (except for $m = 2$ where they coincide). While the proportion of t-local equilibria tends to decrease when the number of candidates increases under fixed candidates' preferences, this tendency is not visible under narcissistic candidates' preferences. This can be explained by the fact that candidates may have less freedom to strategize when the hypercube is divided among several candidates' sets of influence: it can be more difficult for a candidate to find enough space for a deviation that would make her win.



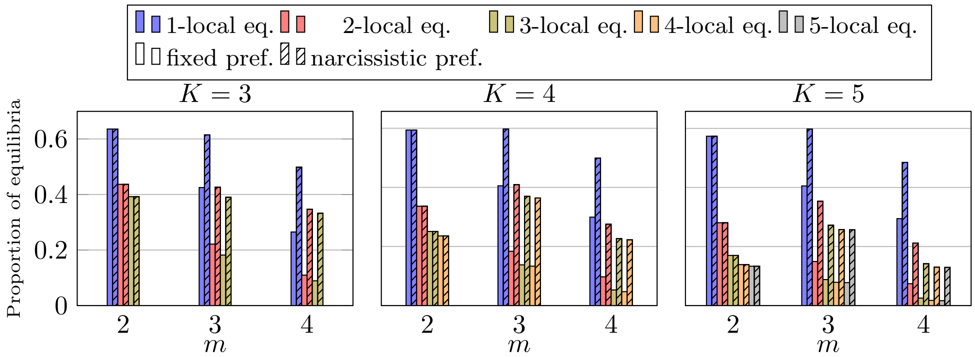
(a) Candidates' strategies: random subsets

(b) Candidates' strategies: random DFS-connected subsets

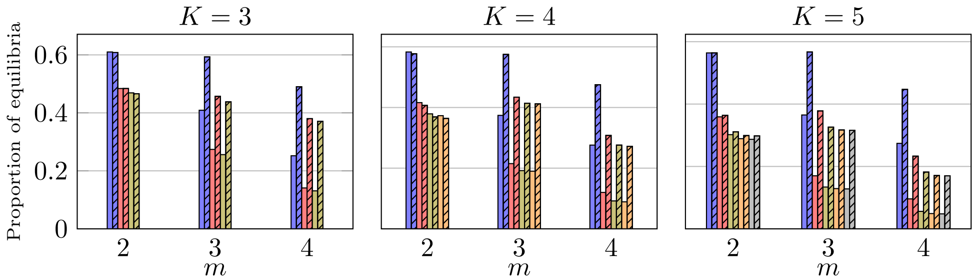
(c) Candidates' strategies: random BFS-connected subsets

(d) Candidates' strategies: random balls

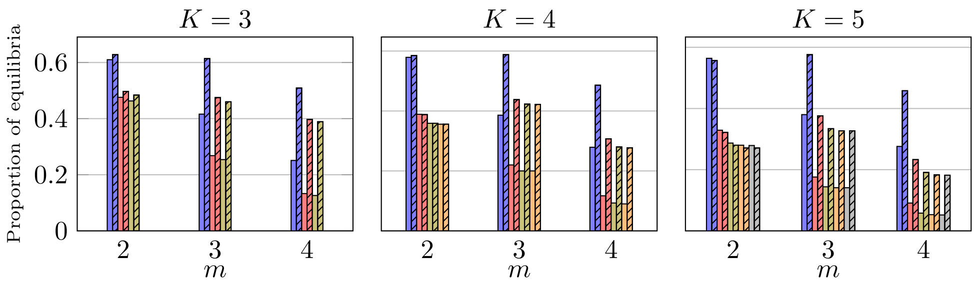
Fig. 1: Proportion of instances where a t-local equilibrium exists; with $m \in \{2, 3, 4\}$ candidates, $K \in \{3, 4, 5\}$ issues, 5,000 voters, and all $t \in [K]$, under fixed or narcissistic candidates' preferences.



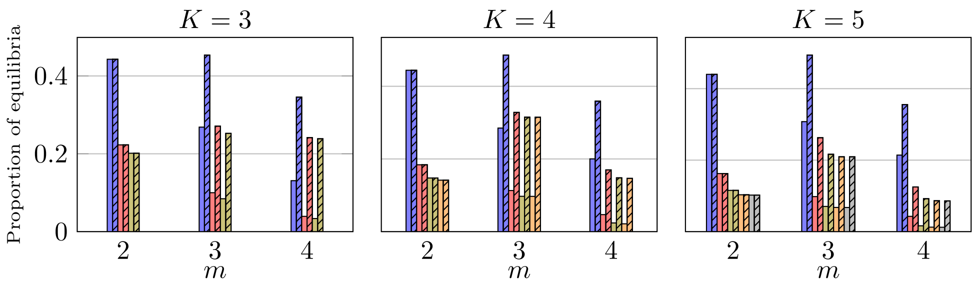
(a) Candidates' strategies: random subsets



(b) Candidates' strategies: random DFS-connected subsets



(c) Candidates' strategies: random BFS-connected subsets



(d) Candidates' strategies: random balls

Fig. 2: Average proportion of states that are t-local equilibria; with $m \in \{2, 3, 4\}$ candidates, $K \in \{3, 4, 5\}$ issues, 5,000 voters, and all $t \in [K]$, under fixed or narcissistic candidates' preferences.

Regarding the different kinds of strategy sets, we can see they have a clear impact over the proportion of states corresponding to each kind of t-local equilibria. For the random connected subsets and random subsets, we observe that the proportion of states corresponding to 1-local equilibria is around 1.5 times the amount from the case of random balls. Similarly, t-local equilibria with $t \in \{2, \ldots, K\}$ correspond to a proportion at least twice as important than for the case of random balls. This could be explained by the fact that randomly generated sets other than balls are generally *sparse*, and thus the set of all possible strategies in the generated game will, in general, be *smaller* (making any proportions seem larger). This sparsity also means that t-local deviations might be unlikely from any given state (e.g., positions in a random/connected $\mathcal{H}_c$ will generally have fewer neighbors in $\mathcal{H}_c$ to which to 1-locally deviate), making equilibria more likely than in the "dense" random balls setting. All in all, these results tell us that under average random conditions, t-local equilibria are really common.

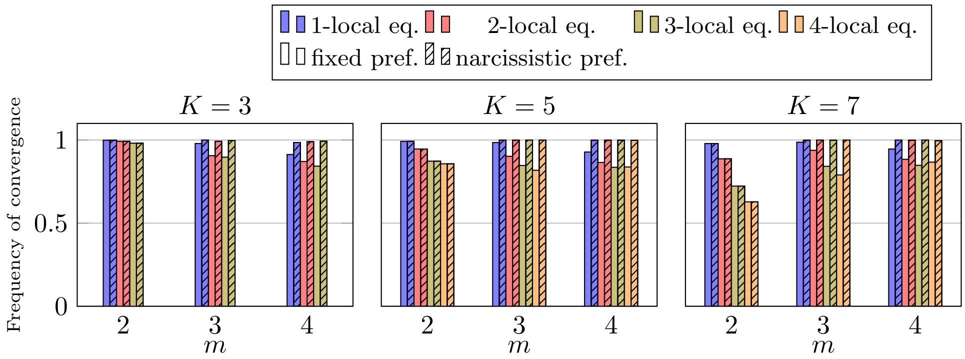
5.2 The Dynamics of Local Deviations

For the experimental study of the dynamics of t-local deviations, we consider successive *rounds* of the game, in which at every given iteration, exactly one player is selected (at random) to choose (at random) any t-local best response she might have from the current state. The initial state of the dynamics is the *truthful* state where every candidate $c \in C$ is placed in her truthful position p^c. Whenever such a simulated dynamic *converges*, it is because a t-local equilibrium is reached; we will say the simulated dynamics are *non-convergent* whenever the sequence of visited states *cycles*, i.e., the dynamic returns to an already visited state.

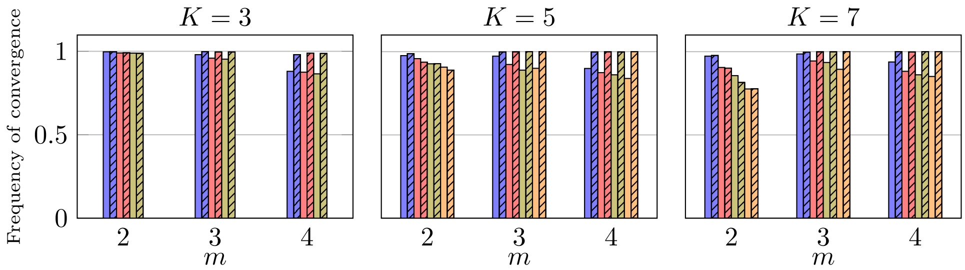
We simulate BSC games for a number of issues $K \in \{3, 5, 7\}$ and a number of candidates $m \in \{2, 3, 4\}$. For each set of parameters, the proportion of games from which the simulated dynamics reach a t-local equilibrium (for $t \in \{1, 2, 3, 4\}$) is represented in Figure 3.

Similarly to our results for the existence, in most cases, t-local equilibria can be reached by randomly following an improving move dynamic from the truthful state. Note nevertheless that, for some parameters, around 20% of the cases, we do stumble upon *cycles* in the dynamics. It seems like 1-local dynamics have a higher tendency towards reaching 1-local equilibria, than the rest of dynamics, which would be in line with the fact that 1-local equilibria are more frequent than other t-local equilibria (see Figure 2). On a similar vein, under narcissistic preferences, the dynamics tend to converge in almost every scenario (especially for $m > 2$). This is consistent with our observations about how common t-local equilibria are under narcissistic preferences; and also that, for bigger m, the division of the hypercube among the different candidates' influence sets makes it hard for any individual candidate to directly strategize to win the election.

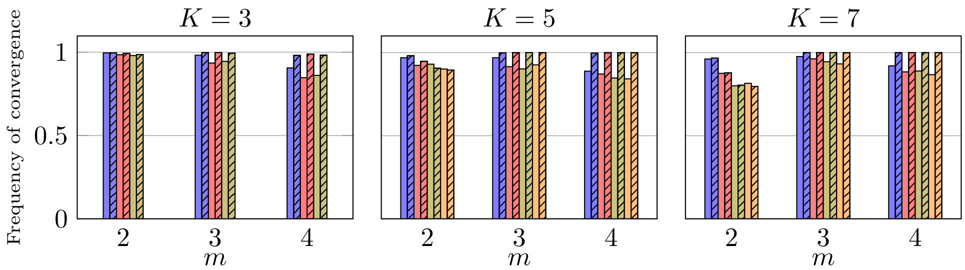
Figure 4 displays, for all the different configurations of our parameter space, the number of *iterations* (or turns) that are required for the dynamics to converge. It is seen, as expected, that under fixed candidates' preferences, for a greater number of candidates the amount of deviations required to reach a stable state is significantly larger. This is, of course, due to the greater amount of possible candidates who might



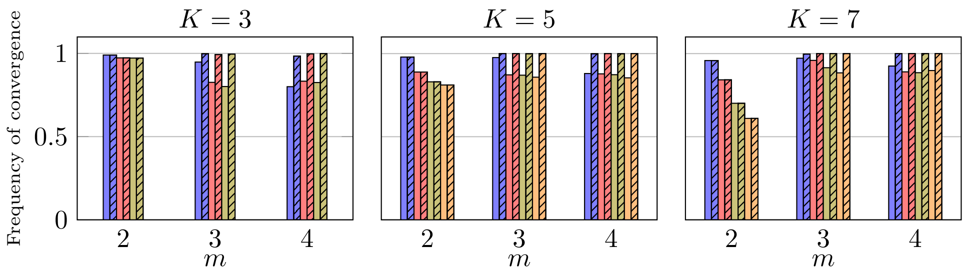
(a) Candidates' strategies: random subsets



(b) Candidates' strategies: random DFS-connected subsets

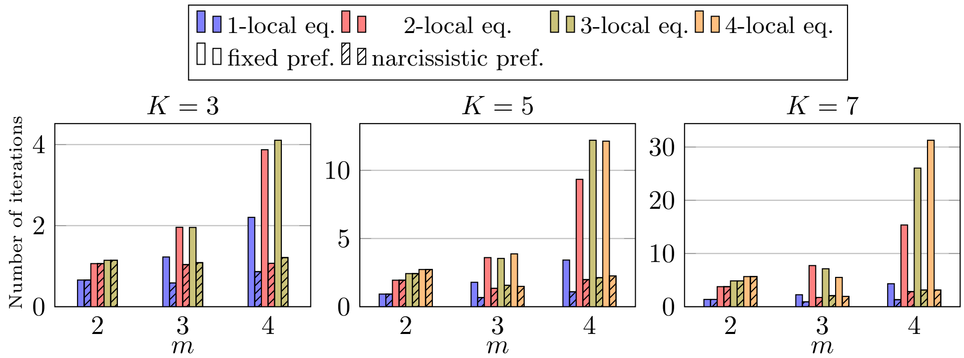


(c) Candidates' strategies: random BFS-connected subsets

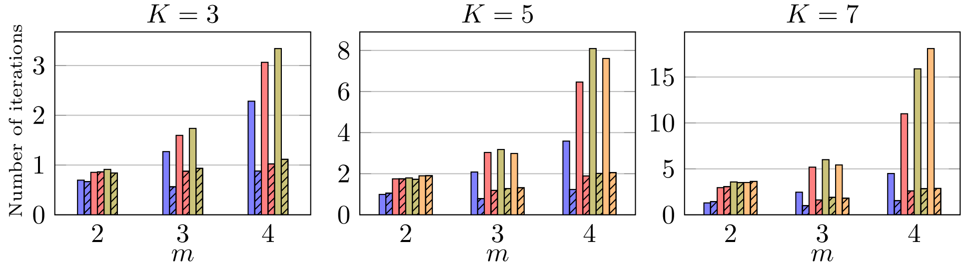


(d) Candidates' strategies: random balls

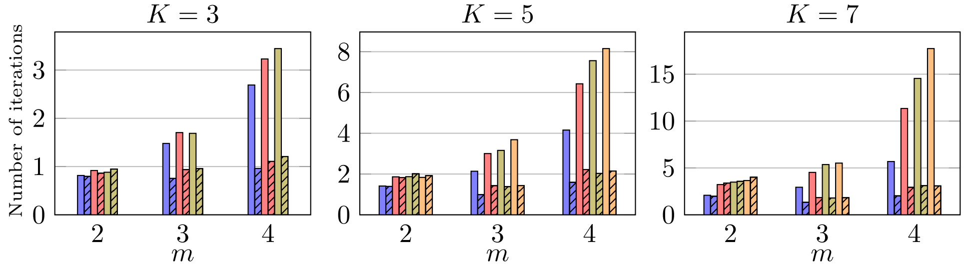
Fig. 3: Average proportion of the t-local dynamics that reach a t-local equilibrium from the initial *truthful* state; with $m \in \{2, 3, 4\}$ candidates, $K \in \{3, 5, 7\}$ issues, 5,000 voters, and $t \in \{1, 2, 3, 4\}$, under fixed or narcissistic candidates' preferences.



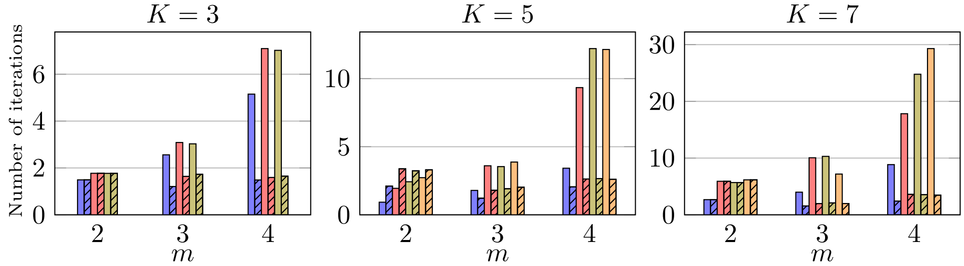
(a) Candidates' strategies: random subsets



(b) Candidates' strategies: random DFS-connected subsets

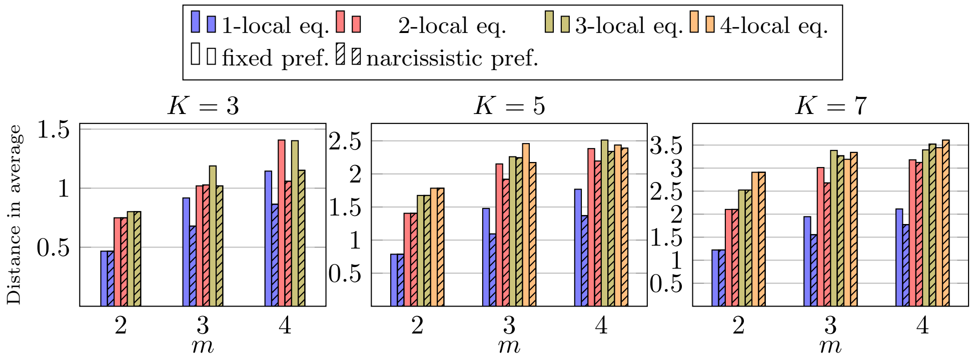


(c) Candidates' strategies: random BFS-connected subsets

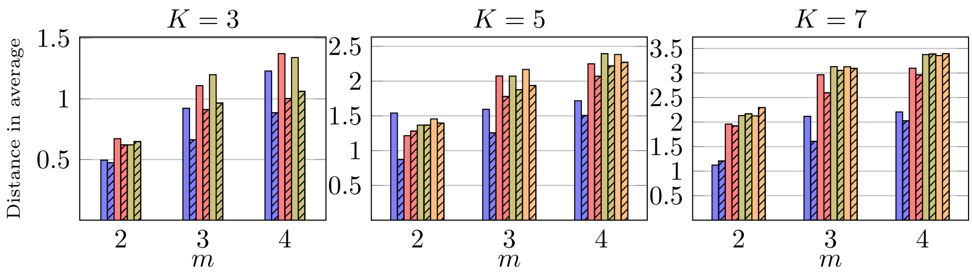


(d) Candidates' strategies: random balls

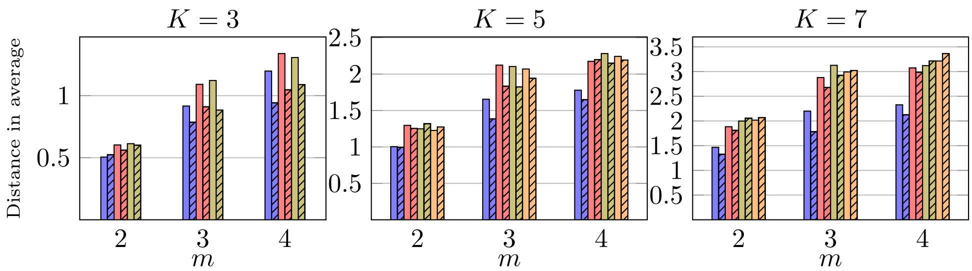
Fig. 4: Average number of iterations before convergence to a t-local equilibrium; with $m \in \{2, 3, 4\}$ candidates, $K \in \{3, 5, 7\}$ issues, 5,000 voters, and $t \in \{1, 2, 3, 4\}$, under fixed or narcissistic candidates' preferences.



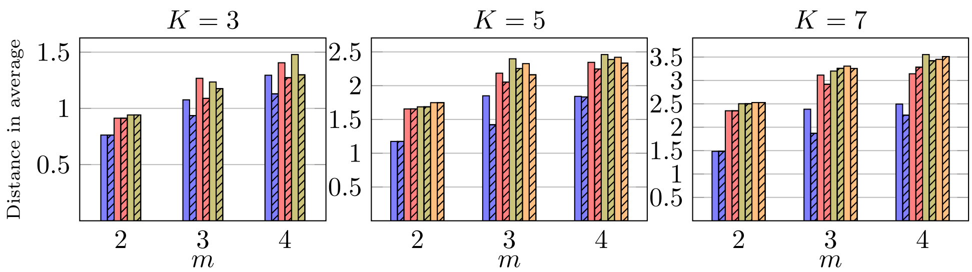
(a) Candidates' strategies: random subsets



(b) Candidates' strategies: random DFS-connected subsets



(c) Candidates' strategies: random BFS-connected subsets



(d) Candidates' strategies: random balls

Fig. 5: Average distance between the initial winner's displayed strategy and the winner's displayed strategy at the t-local equilibrium reached by the t-local dynamics; with $m \in \{2, 3, 4\}$ candidates, $K \in \{3, 5, 7\}$ issues, 5,000 voters, and $t \in \{1, 2, 3, 4\}$, under fixed or narcissistic candidates' preferences.

have an improving deviation. We notice that the number of iterations required to converge increases both with the number of candidates and issues; however this increase is not as significant for the 1-local dynamics as for the other ones (which all behave in a somewhat similar manner). Once again, this may be explained by the fact that 1-local equilibria are significantly more frequent, and thus they might be found *faster* within the dynamics.

This argument about a large proportion of states being t-local equilibria implying faster convergence of the dynamics can be applied for several other interesting factors. For instance, under narcissistic preferences, the same argument can be used to explain the few steps required to converge for any of the t-local dynamics. Analogously, the smaller number of iterations required to converge under random or random connected strategy sets (as compared to the random balls) is similarly explained thanks to our analysis of Figure 2.

Finally, we aim to study some metric to assess 'how far the state reached by the dynamic was from the original, fully truthful, state'. We thus considered the *distance* between the winner's position in the initial and end states. Figure 5 shows the average distance for each considered combination of parameters. Despite the random nature of our simulations, this metric should allow us to identify, at least in the average case, whether the t-local dynamic produces *radically different winners* from the ones in the original, *truthful*, profiles. A deeper theoretical study of this metric under t-local dynamics, as well as some possible links to the Price of Anarchy and other related concepts, is undoubtedly an interesting line of work to be tackled in the future.

Observing Figure 5, we clearly see that, given a fixed number of issues, as we increase the number of candidates in the game (and thus, the possibilities of deviating), the average distance gets closer and closer to $\frac{K}{2}$. In some sense, for those cases the end position is as good as if it had been chosen uniformly at random (in which case we would see a distance of $\frac{K}{2}$ in expectation). We also notice that this increase towards $\frac{K}{2}$ is significantly slower for narcissistic preferences (in general), and also for random or random connected strategy sets as compared to the random balls. As before, this might be due to the higher proportion of states corresponding to equilibria, as it means that, often, the dynamics will directly start at an equilibrium state (an thus the dynamics will make no deviations at all).

In general (but more remarkably for the random balls and random sets) the 1-local dynamics converge to states whose distance (between initial and final winners) is significantly smaller than for the dynamics with larger t. This can be explained, again, by the large proportion of states corresponding to 1-local equilibria (under all settings) as compared to other $t \in \{2, \ldots, K\}$.

A final illustration of the same argument is the case of $m = 2$ candidates (under any strategy sets), for which consistently the final winner does not drift too far away from the original one (again, due to the high number of equilibria for $m = 2$).

A general takeaway from our experimental setup (of the dynamics) is that a larger proportion of a given type of t-local equilibrium will lead to more *robust* BSC dynamics in general. This means that the associated t-local dynamics will generally converge not only *faster*, but also to a new winner that will not be dramatically far away (in displayed position) from the original truthful winner.

6 Conclusion

We have introduced a Hotelling-Downs game to capture the strategic behavior of candidates that may lie about their true opinions in an election. Beyond the classical left-right axis, we have proposed to model political views via binary opinions over issues, leading to work with a very structured environment, i.e., the hypercube. In this context, a natural notion of distance arises, giving birth to the solution concept of local equilibrium. While in general local equilibria may not exist, we have identified several meaningful conditions under which the existence is guaranteed. Moreover, our experimental results balance the apparently negative theoretical results since equilibria almost always exist in practice, and can be mostly reached by successive local deviations. All our findings highlight a very interesting behavior for t-local equilibria: it seems that there is a clear frontier for positive results between $t = 1$ and the rest. Since 1-local deviations are the most realistic moves, this suggests that the outcome of an election with strategic candidates may not be disastrous: the election would stabilize rather quickly on an equilibrium, electing a candidate not that far from the sincere outcome.

Our work opens several interesting and challenging questions. First of all, there are still some gaps in our theoretical results that would be worth investigating. In particular, does a 1-local equilibrium always exist under narcissistic preferences for $m \geq 3$ candidates, as our experiments suggest? Similarly, it may be of interest to consider voting rules other than plurality when $m \geq 3$. In our specific setting on binary issues, aggregation rules from *Judgment Aggregation* [27] would be particularly relevant, think, e.g., about the classical majority rule which takes the majoritarian outcome on each issue independently. Integrating withdrawal as an additional possible strategy for candidates or assuming that both voters and candidates are strategic (see, e.g., [17]) are also immediate extensions of our model.

A model even closer to that of Harrenstein et al. [7] and to the setting of *Voronoi games on graphs* would be one on which candidates choose to deviate if they are able to *increase the amount of votes that they receive* (without necessarily winning the election). Such lane of study certainly seems like an interesting development to consider. This would nevertheless take us away from the original idea of strategic candidacy where candidates may choose to favor other candidates if they cannot be elected themselves.

Finally, regarding our empirical results, many improvements might be sought. On one hand, with the proper computational resources, broader ranges of parameters for the simulations could be considered. On the other hand, in order to truly validate some of the concepts introduced in the paper, it could be helpful to consider experiments based on *real-world data* of elections with *binary issues* (see, e.g., the Stemwijzer from the Netherlands or the Wahl-O-Mat from Germany, which has a publicly available dataset). Exploring these empirical ideas in depth, could undoubtedly be an interesting line of future work.

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